Chapter 6: Optimization problems

ENSIIE - Computational complexity theory

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Optimization problem

(Informal) definition

An optimization problem is a problem containing inputs (the *instances*), solutions (the *feasible solutions*), a *measure* or *weight* associating to each solution an integer. The objective is to find a solution maximizing or minimizing the measure.

Problems associated with an optimization problem

(Informal) definition

An optimization problem Π is associated with 3 problems:

- a *decision problem* Π_D: does there exist a feasible solution with weight
  - lower than K?, if the objective is to minimize the measure,
  - greater than K?, if the objective is to maximize the measure ;
- an *evaluation problem* Pi_E: find the weight of an optimal solution;
- a *construction problem* Π_C: find an optimal solution and its weight.

PO and NPO

Definition

PO and NPO are the equivalent classes of P and NP for the optimization problems: the optimization problems that can be solved in deterministic or non deterministic polynomial time.

Particularly,

- $\Pi \in PO \Rightarrow \Pi_D \in P$
- $\Pi \in NPO \Rightarrow \Pi_D \in NP$

Optimization problem

Definition: optimization problem

An optimization problem Π is a quadruplet $(\mathcal{I}, \mathcal{S}, \mathcal{M}, \mathcal{O})$ such that:

- $\mathcal{I}$ is the set of d'*instances* of Π ;
- $\mathcal{S}$ is a function associating to $x \in \mathcal{I}$ a set of *feasible solutions* of x ;
- $\mathcal{M}$ is a *measure* function associating to $x \in \mathcal{I}$ and $y \in \mathcal{S}(x)$ an integer ;
- $\mathcal{O}$ is the *objectif* with value min or max that specify whether we want to find a solution with minimum of maximum measure.

$\mathcal{I}$ and $\mathcal{S}(x)$ may contain any mathematical objects.

Optimal solution

Definition

Let $\Pi = (\mathcal{I}, \mathcal{S}, \mathcal{M}, \mathcal{O})$ be an optimization problem and $x \in \mathcal{I}$. We call *optimal solution* a solution $y^* \in \mathcal{S}(x)$ such that:

$$\mathcal{M}(x, y^*) = \min_{y \in \mathcal{S}(x)} \mathcal{M}(x, y) \text{ si } \mathcal{O} = \min$$

$$\mathcal{M}(x, y^*) = \max_{y \in \mathcal{S}(x)} \mathcal{M}(x, y) \text{ si } \mathcal{O} = \max$$

We denote the *set of optimal solutions* of x by $\mathcal{S}^*(x)$ and the value $\mathcal{M}(x, y^*)$ of the optimal solutions by $\mathcal{M}^*(x)$.

Size of an instance or of a solution

Definition

The size of an instance or of a feasible solution is the number of bits used to encode it.

We usually denote the size of the input x by $|x|$ or n and the size of a feasible solution y by $|y|$.

Problems associated with an optimization problem

(Informal) definition

An optimization problem $\Pi = (\mathcal{I}, \mathcal{S}, \mathcal{M}, \mathcal{O})$ is associated with 3 problems:

- a *decision problem* Π_D: let $x \in \mathcal{I}$ and an integer K, determine if $\mathcal{M}^*(x) \leq K$ if $\mathcal{O} = \min$, or if $\mathcal{M}^*(x) \geq K$ if $\mathcal{O} = \max$.
- an *evaluation problem* Pi_E: let $x \in \mathcal{I}$, compute $\mathcal{M}^*(x)$;
- a *construction problem* Π_C: let $x \in \mathcal{I}$, compute an optimal solution $y \in \mathcal{S}^*(x)$ and $\mathcal{M}^*(x)$.

Some results

Theorem

Let Π be an optimization problem:

$$\Pi_D \preccurlyeq_T \Pi_E \preccurlyeq_T \Pi_C$$

NPO

Definition

Let $\Pi = (\mathcal{I}, \mathcal{S}, \mathcal{M}, \mathcal{O})$ be an optimization problem, Π belong to the NPO class if

- let x be a binary number, we can check in polynomial time if x encodes an instance of $\mathcal{I}$;
- there exists a polynom q such that for every $x \in \mathcal{I}$
  - for every $y \in \mathcal{S}(x)$, $|y| \leq q(|x|)$,
  - for every binary number y such that $|y| \leq q(|x|)$, we can check in polynomial time if y encodes a feasible solution of x;
- for every $x \in \mathcal{I}$ and every $y \in \mathcal{S}(x)$, we can compute $\mathcal{M}(x, y)$ in polynomial time.

Definition

A problem $\Pi \in$ NPO belongs to the class PO if we can solve Π_C in polynomial time.

By definition PO $\subseteq$ NPO.

PO, NPO, P and NP

Theorem

$$\Pi \in PO \Rightarrow \Pi_D \in P$$

$$\Pi \in NPO \Rightarrow \Pi_D \in NP$$

$$\Pi \in NPO \Rightarrow \Pi_E \preccurlyeq_T \Pi_D$$

NP-Hard optimization problem

Definition

Let Π_1 be an optimization problem, Π_1 is NP-Hard, for every decision problem $\Pi_2 \in$ NP, $\Pi_2 \preccurlyeq_T \Pi_1$.

Theorem

Let $\Pi \in$ NPO, if Π_D is NP-Hard, then Π is NP-Hard.

Theorem

PO $=$ NPO $\Rightarrow$ P $=$ NP

A hard proof

Theorem

Let $\Pi \in$ NPO, if Π_D is NP-Complete, then $\Pi_C \preccurlyeq_T \Pi_D$.