Tutorial 3 : Reduction and completeness

Computational complexity theory, 5th semester.

2017-2018

Exercice 1 — ***Simple reductions***

1. Prove that (SIZE) $\preccurlyeq$ (INS).
2. Prove that (SIZE) $\preccurlyeq$ (CHROMA).
3. Prove that (INS) $\preccurlyeq$ (W-INS).
4. Prove that (3-COL ?) $\preccurlyeq$ (SAT).
5. Prove that (COHAM) $\preccurlyeq$ (TAU).
6. Prove that (HAM ?) $\preccurlyeq$ (LOP).
7. Prove that (HAM ?) $\preccurlyeq$ (TSP).
8. Prove that (SAT ?) $\preccurlyeq$ (3-SAT ?).
9. Prove that (TAU ?) $\preccurlyeq$ (3-TAU ?).
10. Prove that (SAT ?) $\preccurlyeq$ (QBF ?).
11. Prove that (TAU ?) $\preccurlyeq$ (QBF ?).
12. Prove that (CONNECTIVITY ?) $\preccurlyeq$ (MSPT).
13. Prove that (SUDOKU ?) $\preccurlyeq$ (CHROMA).
14. Prove that (INS) $\preccurlyeq$ (ILP).
15. Prove that (BIPARTI ?) $\preccurlyeq$ (2-COL ?).
16. Prove that (2-COL ?) $\preccurlyeq$ (BIPARTI ?).
17. Prove that (SUBSET SUM ?) $\preccurlyeq$ (PARTITION).
18. Prove that (SUBSET SUM ?) $\preccurlyeq$ (KNAPSACK).
19. Prove that (SET COVER) $\preccurlyeq$ (DST).
20. Prove that (SET COVER) $\preccurlyeq$ (UST).

We know that (SAT) is NP-Complete and (TAU) is Co-NP-Complete. For which of the previous problems can you affirm that they are (Co-)NP-Complete or (Co-)NP-Complete ?

Exercice 2 — ***(SUBSET SUM is NP-Complete)***

On rappelle que (SUBSET SUM) est le problème suivant : soit Y un ensemble d'entiers et $s \in \mathbb{N}$, existe-t-il un sous-ensemble Z de Y dont la somme fait s ?

We want to prove that (SUBSET SUM) is NP-Complete.

1. Show that this problem is in NP.
2. Let $\mathcal{I}$ be an instance of (3-SAT ?), we want to transform $\mathcal{I}$ into an instance $\mathcal{I}$ of (SUBSET SUM ?) in polynomial time such that $\mathcal{I}$ is positive if and only if $\mathcal{J}$ is positive. We call Y the set of integers of $\mathcal{J}$ and s the sum we want to reach.
   - We assume that $\mathcal{I}$ has n variables $x_1, \dots, x_n$ and m clauses $C_1, C_2, \dots, C_m$.
   - The numbers of the instance $\mathcal{J}$ have $2n + 2m$ figures (and are then between 0 and 10^{n+m}).
   - For each variable x_i of $\mathcal{I}$, we add an integer x_i to $\mathcal{J}$ where the i-th figure is 1 and where the $n + j$-th figure is 1 if x_i is in the j-th clause of $\mathcal{I}$. Every other figure is 0.

— For each variable x_i of $\mathcal{I}$, we add an integer $\bar{x_i}$ to $\mathcal{J}$ where the i-th figure is 1 and where the $n+j$-th figure is 1 if $\bar{x_i}$ is in the j-th clause of $\mathcal{I}$. Every other figure is 0.

— For each clause C_j of $\mathcal{I}$, we add two equal numbers r_j and s_j to $\mathcal{J}$ where the $n+j$-th figure is 1. Every other figure is 0.

— s is the number where the n first figures are 1 and the m others are 3.

(a) Describe $\mathcal{J}$ if $\mathcal{I} = (\bar{x_1} \vee x_2 \vee x_3) \wedge (\bar{x_1} \vee x_2 \vee \bar{x_3}) \wedge (x_1 \vee \bar{x_2} \vee \bar{x_3}) \wedge (x_1 \vee \bar{x_2} \vee \bar{x_3})$.

(b) Describe $\mathcal{J}$ if $\mathcal{I} = (x_1 \vee x_2) \wedge (\bar{x_1} \vee x_2) \wedge (x_1 \vee \bar{x_2}) \wedge (\bar{x_1} \vee \bar{x_2})$.

(c) Show that the complexity of the transformation is polynomial.

(d) Using the two examples, show that, if $\mathcal{I}$ is a positive instance, then $\mathcal{J}$ is a positive instance.

(e) We assume that $\mathcal{J}$ is positive, we want to prove that $\mathcal{I}$ is positive too. There exists a subset $Z \subset Y$ such that the sum of the elements of Z is s.

i. Show that $x_i \in Z \Leftrightarrow \bar{x_i} \notin Z$.

ii. Let $C_j = (l_1 \vee l_2 \vee l_3)$ be a clause of $\mathcal{I}$, show that the integer l_1 or l_2 or l_3 is in X.

iii. Deduce that $\mathcal{I}$ can be satisfied.

3. Deduce from the previous question that (SUBSET SUM) is NP-Complete.
4. For which of the problems of Exercise 1 can you affirm that they are NP-Complete or NP-Hard ?

Exercice 3 — ***(SET COVER is NP-Complete)***

On rappelle que (SUBSET SUM) est le problème suivant : soit X un ensemble, S un ensemble de sous-ensembles de X et $K \in \mathbb{N}$, existe-t-il un sous-ensemble C de S de taille inférieure à K couvrant X ? (c'est à dire que pour tout $x \in X$, il existe $s \in C$ tel que $x \in s$).

We want to prove that (SET COVER) is NP-Complete.

1. Show that this problem is in NP.
2. Let $\mathcal{I}$ be an instance of (3-SAT ?), we want to transform $\mathcal{I}$ into an instance $\mathcal{I}$ of (SET COVER) in polynomial time such that $\mathcal{I}$ is positive if and only if $\mathcal{J}$ is positive. We call X the set of elements of $\mathcal{J}$, S the set of subsets of X and K the number of sets we can use.

— We assume that $\mathcal{I}$ has n variables $x_1, \ldots, x_n$ and m clauses $C_1, C_2, \ldots, C_m$.

— X will contain $n+m$ elements $e_1, e_2, \ldots, e_{n+m}$ and S with contains $2n$ sets.

— For each variable x_i of $\mathcal{I}$, we add to S a set x_i and a set $\bar{x_i}$ containing e_i.

— If x_i is in C_j, add e_{n+j} to the set x_i.

— If $\bar{x_i}$ is in C_j, add e_{n+j} to the set $\bar{x_i}$.

— $K = n$.

(a) Describe $\mathcal{J}$ if $\mathcal{I} = (\bar{x_1} \vee x_2 \vee x_3) \wedge (\bar{x_1} \vee x_2 \vee \bar{x_3}) \wedge (x_1 \vee \bar{x_2} \vee \bar{x_3}) \wedge (x_1 \vee \bar{x_2} \vee \bar{x_3})$.

(b) Describe $\mathcal{J}$ if $\mathcal{I} = (x_1 \vee x_2) \wedge (\bar{x_1} \vee x_2) \wedge (x_1 \vee \bar{x_2}) \wedge (\bar{x_1} \vee \bar{x_2})$.

(c) Show that the complexity of the transformation is polynomial.

(d) Using the two examples, show that, if $\mathcal{I}$ is a positive instance, then $\mathcal{J}$ is a positive instance.

(e) We assume that $\mathcal{J}$ is positive, we want to prove that $\mathcal{I}$ is positive too. There exists a subset $C \subset S$ such that each element of X is in at least one set of C and C contains at most K sets.

i. Show that $x_i \in C \Leftrightarrow \bar{x_i} \notin C$.

ii. Let $C_j = (l_1 \vee l_2 \vee l_3)$ be a clause of $\mathcal{I}$, show that $l_1 \in C$ or $l_2 \in C$ or $l_3 \in C$.

iii. Deduce that $\mathcal{I}$ can be satisfied.

3. Deduce from the previous question that (SET COVER) is NP-Complete.
4. For which of the problems of Exercise 1 can you affirm that they are NP-Complete or NP-Hard ?

Exercice 4 — *(CHROMA is NP-Complete)*

On rappelle que (CHROMA) est le problème suivant : soit $G = (V, E)$ un graphe et $K \in \mathbb{N}$, $K \leq |V|$, peut-on colorier V avec K couleurs de sorte que deux nœuds voisins dans G n'aient pas la même couleur.

We want to prove that (CHROMA) is NP-Complete.

1. Show that this problem is in NP.
2. Let $\mathcal{I}$ be an instance of (3-SAT ?), we want to transform $\mathcal{I}$ into an instance $\mathcal{I}$ of (CHROMA) in polynomial time such that $\mathcal{I}$ is positive if and only if $\mathcal{J}$ is positive. We call $G = (V, E)$ the graph of $\mathcal{J}$ and K the number of colors we can use.
   - We assume that $\mathcal{I}$ has n variables $x_1, \ldots, x_n$ and m clauses $C_1, C_2, \ldots, C_m$.
   - G will contain $3n + m + 1$ nodes.
   - Add to G a clique of $n + 1$ nodes $y_1, y_2, \ldots, y_{n+1}$.
   - For each variable x_i of $\mathcal{I}$, we add to V a node x_i and a node $\bar{x_i}$.
   - For each clause C_j of $\mathcal{I}$, we add a node C_j to V.
   - We link x_i to $\bar{x_i}$.
   - If $i \neq j$ and $j < n + 1$, we link x_i to y_j and $barx_i$ to y_j.
   - We link y_{n+1} to C_j for every j.
   - If x_i is not in C_j, link x_i and C_j.
   - If $\bar{x_i}$ is not in C_j, link $\bar{x_i}$ and C_j.
   - $K = n + 1$.

   (a) Describe $\mathcal{J}$ if $\mathcal{I} = (\bar{x_1} \vee x_2 \vee x_3) \wedge (\bar{x_1} \vee x_2 \vee \bar{x_3}) \wedge (x_1 \vee \bar{x_2} \vee \bar{x_3}) \wedge (x_1 \vee \bar{x_2} \vee \bar{x_3})$.

   (b) Describe $\mathcal{J}$ if $\mathcal{I} = (x_1 \vee x_2) \wedge (\bar{x_1} \vee x_2) \wedge (x_1 \vee \bar{x_2}) \wedge (\bar{x_1} \vee \bar{x_2})$.

   (c) Show that the complexity of the transformation is polynomial.

   (d) Using the two examples, show that, if $\mathcal{I}$ is a positive instance, then $\mathcal{J}$ is a positive instance.

   (e) We assume that $\mathcal{J}$ is positive, we want to prove that $\mathcal{I}$ is positive too. There exists a coloration of G with at most K colors. Let c_v be the color of the node v.

      i. Show that $c_{x_i} = c_{y_i}$ and $c_{\bar{x_i}} = c_{y_{n+1}}$ or $c_{x_i} = c_{y_{n+1}}$ and $c_{\bar{x_i}} = c_{y_i}$.

      ii. Let $C_j = (l_1 \vee l_2 \vee l_3)$ be a clause of $\mathcal{I}$, show that $c_{C_j} = c_{l_1}$ or $c_{C_j} = c_{l_2}$ or $c_{C_j} = c_{l_3}$.

      iii. Deduce that $\mathcal{I}$ can be satisfied.
3. Deduce from the previous question that (CHROMA) is NP-Complete.
4. For which of the problems of Exercise 1 can you affirm that they are NP-Complete or NP-Hard ?

Exercice 5 — *Some proof*

1. Show that the reduction relation is transitive.
2. If we assume that there exists a problem $\Pi = (\mathcal{L}, \mathcal{L}_Y, \mathcal{L}_N)$ such thatΠ and $\Pi^c = (\mathcal{L}, \mathcal{L}_N, \mathcal{L}_Y)$ are NP-Complete, then show that NP = Co-NP.
3. An oracle for Π is defined as a machine that solve in constant time the problem Π. What would happen if we have an oracle for (3-SAT ?) ?

Exercice 6 — *Turing reduction*

1. Show that the polynomial reduction of Karp is a special case of polynomial Turing reduction.
2. Show that, for every problem Π in NP, there exists a problem of Co-NP such that there exists a polynomial Turing reduction from that problem to Π, and conversely.
3. Deduce that NP and Co-NP are equivalent if we use the polynomial Turing reduction.
4. Show that, if NP $\neq$ Co-NP, then there does not exist any Co-NP problem which is NP-hard and conversely if we use the polynomial Karp reduction.