Tutorial 5 : Inputs encoding

Computational complexity theory, 5th semester.

2022

Exercice 1 — *Weakly and strongly NP-Complete*

For each of the NP-Complete problems of the first tutorial, determine if it is strongly NP-Complete or weakly NP-Complete.

Exercice 2 — *Some results*

1. What prove a reduction from a weakly NP-Complete problem ?
2. Show that a polynomial Karp reduction from a strongly NP-Complete problem Π_1 to a problem Π_2 does not prove that Π_2 is strongly NP-Complete.
3. Prove that, if a problem is strongly NP-Complete (which means that it is NP-Complete if its integers are unary encoded), the problem is still NP-Complete if its integers are binary encoded.

Exercice 3 — *Weakly and strongly Polynomial*

1. Is the simplex algorithm solving linear programs strongly polynomial ? We now restrict the instances to programs with at most 2 variables, is the simplex algorithm, in that case, strongly polynomial ?
2. Describe a strongly polynomial algorithm to compute the mean of n numbers.
3. We want to compute the n-th digit of 2 with the Newton and Raphson method.
   — Let f be a strictly convex and increasing function on $D \subset \mathbb{R}$, we search for a real $x \in D$ such that $f(x) = 0$. (We assume that such a real exists). Let $x_0 \in D$ and $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$, show that the sequence $(x_n)_{n\in\mathbb{N}}$ is convergent and approaches x.
   — Show that $x_n - x \leq f(x_n)/f'(x)$
   — Let $f(x) = x^2 - 2$. Describe an algorithm to compute the n-th digit of $\sqrt{2}$.
   — Show that, if $x_0 < \sqrt{2} + 0.3$,the algorithm stops in n iterations. Deduce the complexity of the algorithm. Is it pseudo polynomial, strongly polynomial or weakly polynomial ?

Exercice 4 — *Succinct graph* A boolean circuit C is a directed graph with no circuit where the n sources are the inputs (1 or 0), the sink is unique and is an output (1 or 0 too) and the nodes are logical operators AND, OR and NOT. For an input $x \in 0, 1^n$, we write $C(x)$ the associated output.

Let G be a undirected graph with 2^n nodes numbered from 1 to 2^n. We define the succinct representation of G as a boolean circuit C with $2n$ inputs (in other words two binary numbers x and y of size n) such that, for every couple of nodes number x and y, the output $C(x, y) = 1$ if and only if x and y are linked by an edge. Warning : not all the graphs have a succinct representation.

We want to show that the following problem is NP-Complete.

(TRIANGLE-SUCC ?) : let C be a boolean circuit representing a graph G, does G have a triangle (a clique of size 3) ?

1. Show that (TRIANGLE-SUCC ?) belongs to NP.
2. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ such that $V_1 \subset V_2$ and such that those two graphs have a succinct representation, show that $G = (V_2, E_1 \cup E_2)$ has a succinct representation.

3. Let φ be an instance of (SAT ?) with n variables $x_1, x_2, \ldots, x_n$. Let $G_\varphi = (V_\varphi, E_\varphi)$ the following graph. V_φ contains $2^n + 1$ nodes : $(v_0, v_1, \ldots, v_{2^n})$. We link v_i to v_{2^n} in E_φ if and only if, when TRUE is affected to x_j if the j-th bit of the binary representation of i is 1 and FALSE otherwise, φ is satisfied. Show that G_φ has a succinct representation.
4. Let $G_2 = (V_2, E_2)$ be the graph containing $2^n + 2$ nodes $(v_0, v_1, \ldots, v_{2^n+1})$ and where every node is linked to v_{2^n+1} (except v_{2^n+1} itself). Show that G_2 has a succinct representation.
5. Deduce that $G = (V_2, E_\varphi \cup E_2)$ has a succinct representation.
6. Show that G has a triangle if and only if φ can be satisfied.
7. Deduce that (TRIANGLE-SUCC) is NP-Complete.