Tutorial 2 : Transitive closure

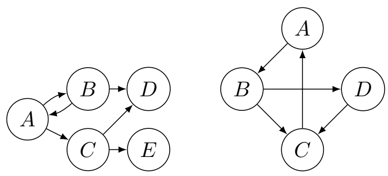
Graph theory, 1st semester.

2022

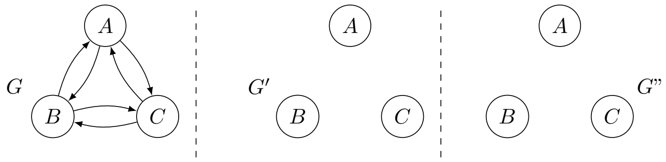
Exercise 1 — *Some examples and properties*

Two graphs G and G' are **τ-equivalent** if $\tau(G) = \tau(G')$. A partial graph G' of G is **τ-minimal τ-équivalent** to G if it is τ-equivalent to G and, if we remove an arc of G', we get a graph that is not τ-equivalent to G. A graph G' is **τ-minimum τ-équivalent** to G if it is **τ-minimal τ-équivalent** to G and minimizes the number of arcs.

1. Draw the transitive closure of the following graphs :



2. Add some arcs to G' and G" such that G' and G" are τ-minimal τ-equivalent to G and G" contains strictly less arcs than G'.



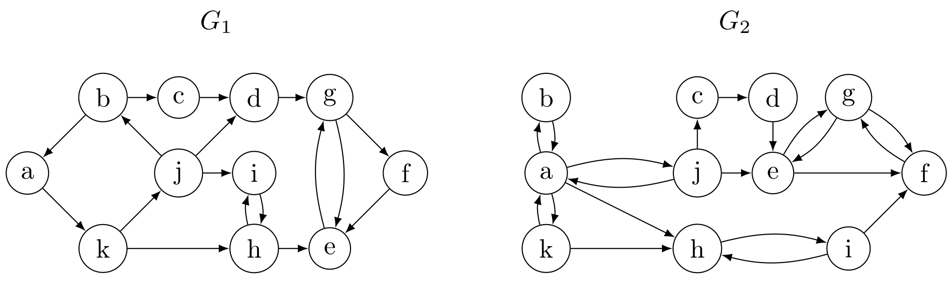
3. Show that G is strongly connected if and only if the circuit $(x_1, x_2, \ldots, x_n)$ is τ-equivalent to G, where the x_is are the nodes of G.
4. Show that the previous property is false if we replace τ-equivalent by τ-minimal τ-equivalent.
5. Show that an elementary circuit is τ-minimum τ-equivalent to itself.
6. Show that a graph G is hamiltonian if and only if, every graph τ-minimum τ-equivalent to G is an hamiltonian circuit of G.

Exercise 2 — *τ equivalent graph of a acyclic digraph*

Let G be a directed acyclic graph. Show that there exists a unique graph G' that is τ-equivalent to G and that minimize the number of arcs. Show that G' is a partial graph of G.

Exercise 3 — *τ-equivalent graphs and reduced graphs*

Let G_1 and G_2 be the two following graphs :



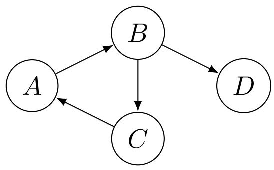
1. Compute the transitive closures of G_1 and G_2.
2. Compute the reduced graphs G_{1R} and G_{2R} of G_1 and G_2.
3. Compute the transitive closures of G_{1R} and G_{2R}.
4. Show that two graphs are τ-equivalent if and only if their reduced graphs are τ-equivalent.

Exercise 4 — *Compute the transitive closure of a graph*

1. Describe with pseudo code an algorithm computing the transitive closure of a graph using the classical algorithms enumerating the nodes of a graph. What is its complexity ?
2. (a) Let A be the adjacency matrix of a graph G. What is A^p ?
   (b) Let B be the adjacency matrix of a graph G with 1 on the diagonal cells. What is B^p ?
   (c) Describe in pseudocode an algorithm computing the transitive closure of a graph using the powers of either the matrix A or the matrix B. What is its complexity ?
3. The Roy Wharshall algorithm is the following :

Require: A directed graph $G = (U, A)$
Ensure: The transitive closure of G
  for $w \in U$ **do**
    for $u \in U$ **do**
      for $v \in U$ **do**
        if $(u, w) \in A$ and $(w, v) \in A$ **then** Add (u, v) to A.

(a) Run the algorithm on the following graph :



(b) What is its complexity ?
(c) Describe the algorithm using only the adjacency matrix of G.
(d) Run the algorithm on the following matrix.

$$\begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Exercise 5 — *Equipment of a workshop*

Nine machines, a, b …, h and i, are installed in a workshop. Mechanical parts are produced by successive machines (drills, polishers, welds, …). We want to build conveyor belts at minimum cost in order to move the pieces from a machine to another. The following array gives, for each machine M, which machines may be used in a next step to build a product. We must link M to each of those machines by a succession of belts. Model this problem with a graph problem and solve it.

a	b, c, d, e, f, g, h, i
b	a, c, d, e, f, g, h, i
c	d, e
d	e
e	d
f	d, e, g, h, i
g	d, e, f, h, i
h	d, e, f, g, i
i	d, e, f, g, h