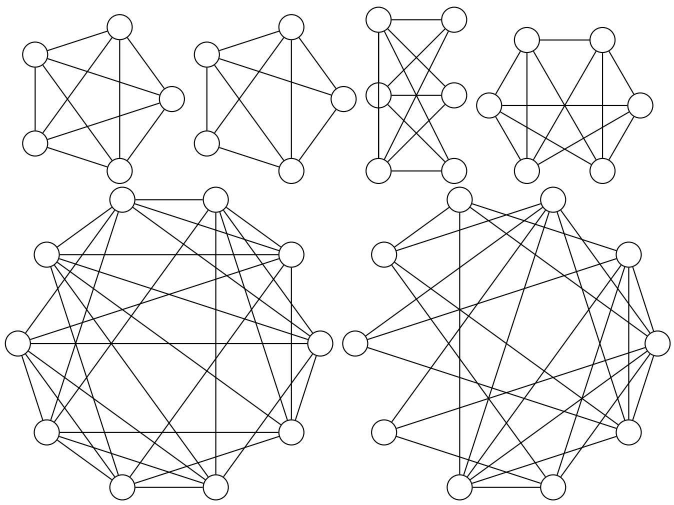
Tutorial 7 : Planar graphs

Graph theory, 1st semester.

2022

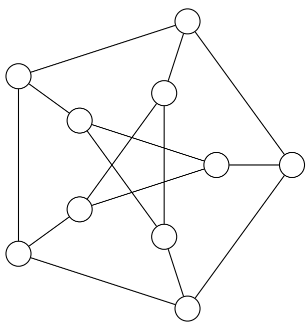
Exercise 1 — *Planar graph ?*

Which of the following graphs are plana graphs ? Check the euler formula for each of the planar graphs and explain why the non planar graphs are not planar.



Exercise 2 — *Petersen graph*

We want to prove the following graph, named the Petersen graph, is not planar.



1. Show that this graph satisfies $m < 3n - 5$.
2. Show every cycle of this graph has 5 edges or more.
3. Let G be a simple connected planar graph with no cycle with c edges or less than c edges, show that $m < \frac{(c+1)(n-2)}{c-1} + 1$.
4. Deduce that the Petersen graph is not planar.

Exercise 3 — *$K_{3,3}$ is planar on a mug*

Show that it is possible to draw $K_{3,3}$ on a torus.

Exercise 4 — *Printed circuit design*

Seven components, (A, B, ..., G) with connection points (from 1 to 4, $a_1, a_2, b_1, b_2, b_3, \dots,$) must be put on a printed circuit. No two connection point has to be linked and no two connection link can cross each other. We have to connect the following poinst
$a_1 - d_1, a_2 - e_1, b_1 - e_2, b_1 - g_2, b_2 - f_1, b_3 - d_2, c_1 - f_2, c_2 - g_1, c_3 - e_3, c_4 - d_3$

1. Show that it is possible to print the circuit.
2. Show that in a bipartite simple connected planar graph, $m < 2n - 3$.
3. Is it possible to add a new connection in the circuit ?

Exercise 5 — *Planar graph and cycle basis*

Let G be a planar graph. We want to show that the internal faces are a cycle basis, by induction on f, the number of faces. Let $F = \{F_1, F_2, \dots, F_{f-1}\}$ be the internal faces of G.

1. Let G be a graph with no face. Show that it does not contain any cycle. Deduce that the property is true when $f = 1$.
2. We now assume that the property is true for any graph with f faces.
   (a) Let e be an edge of the cycle surounding F_1. Show that $G \backslash \{e\}$ is planar. Deduce a cycle basis of that graph.
   (b) Show that we can obtain a cycle basis of G by adding F_1 to the basis of $G \backslash \{e\}$.
   (c) What can we say if all the cycles of the cycle basis of $G \backslash \{e\}$ are internal faces of G? Show a case where this happens.
   (d) Otherwise, show that at most one cycle C of that basis is not an internal face of G?
   (e) Show that $C \oplus F_1$ is an internal face of G.
   (f) Deduce that the internal faces of G are a cycle basis.