Tutorial 8 : Minimum coloration of a graph

Graph theory, 1st semester.

2022

Exercise 1 — *Some bounds*

We write $\chi(G)$ the chromatic number of G.

1. What is the chromatic number of an elementary cycle ? And of a bipartite graph ?
2. Let G_0 be a partial subgraph of G. What relation exists between the chromatic numbers of those graphs ? Deduce that $chi(G) \geq \omega(G)$ where $\omega(G)$ is the size of a maximum clique in G.
3. Let G be a graph of size n and $\alpha(G)$ be the size of a maximum independant set of G. Prouve that :
   — $\chi(G) \cdot \alpha(G) \geq n$
   — $\chi(G) + \alpha(G) \leq n + 1$
4. Show that there exists non-planar graphs that can be colored with 4 colors.

Exercise 2 — *Orientation*

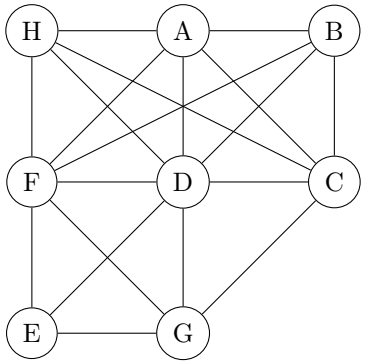
Let G be a graph and k a positive integer. Show that if G can be colored with k colors, then there exists a way to direct every edge into an arc such that G contains no circuit and the longest path is G has size k.

Exercise 3 — *Planar graph*

Describe a simple algorithme that, given a planar graph G, returns an integer k such that $k = \chi(G)$ or $k = \chi(G) + 1$.

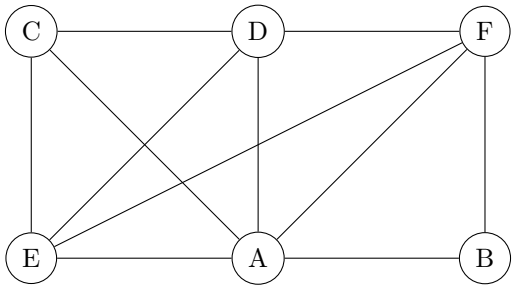
Exercise 4 — *Greedy algorithm*

1. Apply the greedy coloring algorithm of Welsh Powell to the following graph and check that the number of colors is lower or equal to k, the number of the last node satisfying $k \leq d_k + 1$.
2. Can we say this coloration is optimal if we use the bounds you know about the chromatic number ?



Exercise 5 — *Link contract*

Apply the link and contract algorithm to the following graph.



Exercise 6 — *Planning*

Five students must take exams. We want that all the students that have to take a same exam take it at the same time. Each student can only take one exam per day.

This is the list of the exams that every student must take :

— Student 1 : French, English, Mechanics

— Student B : English, Music theory

— Student C : Drawing, History of art, Mechanics

— Student D : Drawing, History of arts

— Music Theory

Model this problem using a node coloration problem. In which cas can we model it with an edge coloration problem ? Deduce that the minimum number of days of exams is either 3 or 4.

Exercise 7 — *Coloration of a Sudoku*

Model a classical Sudoku problem with a coloration problem.