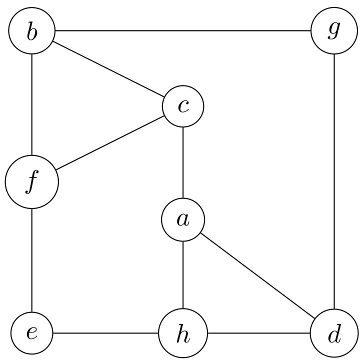
Tutorial 9 : Spanning trees

Graph theory, 1st semester.

2022

Exercise 1 — ***Spanning tree and cycle basis***

Let G be the following graph :



1. Build a spanning tree T of G such that the associated cycle basis is the set of finite faces of G.

Exercise 2 — ***Build a telecommunication network***

A bank wants to build a telecommunication network linking its main agency, situated at the center of Paris, at *Bourse*, and seven of its secondary agencies. The cost needed to connect two agencies is given in the following array :

	B	O	E	R	SL	L	N
Bourse							
Opera	5						
Etoile	18	17					
République	9	11	27				
St-Lazare	13	7	23	20			
Louvre	7	12	15	15	15		
Neuilly	38	38	20	40	40	35	
Chatelet	22	15	25	25	30	10	45

Model this problem with a graph optimization problem and solve it.

Exercise 3 — ***Some tree properties***

1. Show that any tree with two nodes has at least two pendant nodes (with degree 1).
2. Show that any connected graph has two non-articulation nodes.
3. Give a graph with only two non-articulation nodes.

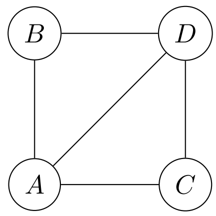
Exercise 4 — ***Connectivity of the spanning trees graph***

Let $G = (V, E)$ be a undirected graph and $\mathcal{T}$ be the set of spanning trees of G. Let $H = (\mathcal{T}, E_H)$ be the graph where

— the nodes are spanning trees of G

— An edge links two nodes of H corresponding to spanning trees T_1 and T_2 of G if and only if all the edges of T_1 and T_2 are the same except for one.

1. Draw H when G is the following graph :



2. Show that H is connected.

Exercise 5 — ***Prim algorithm***

The Prim algorithm finds a minimum spanning tree.

Require: An undirected graph $G = (V, E)$ with weights $\omega : E \to \mathbb{R}^+$.
Ensure: A minimum spanning tree of G

$T = (V_T, E_T) = (\emptyset, \emptyset)$
Add an arbitrary node v to V_T
while $|V_T| \neq |V|$ **do**
    Add to E_T an edge of minimum weight linking $u \in V_T$ to a node $v \in V \backslash V_T$
    Add v to V_T
return T

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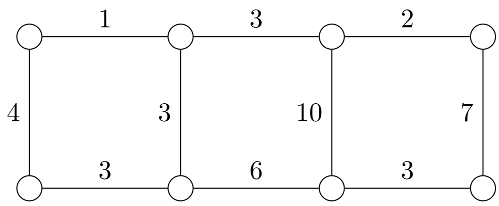
1. Apply the algorithm to the following graph :



2. Let T_i be the tree T at the beginning of iteration i and e be the edge chosen during that iteration. Let $\mathcal{T}_i$ be the set of spanning trees of G covering T_i. Show that there exists a minimum cost tree of $\mathcal{T}_i$ containing e.
3. Deduce that the Prim algorithm is optimal.