Chapter 6 : Penalty and barrier methods

ENSIIE - Operations Research Module

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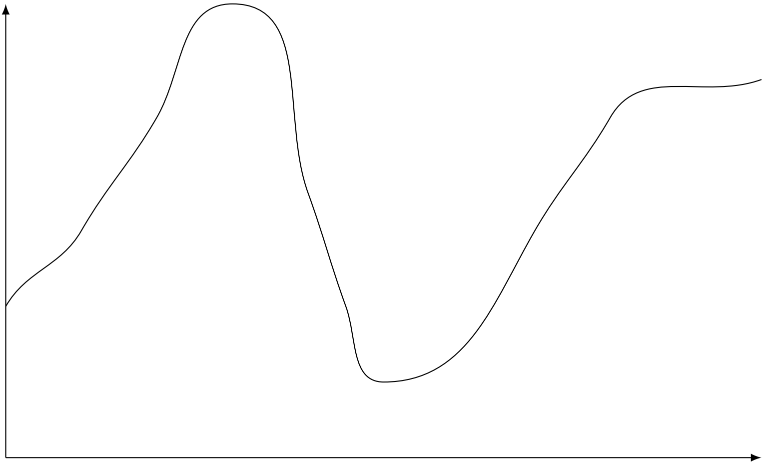
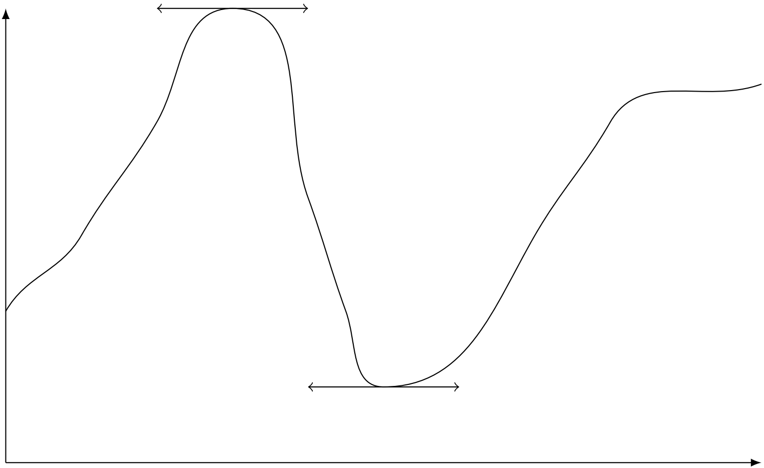
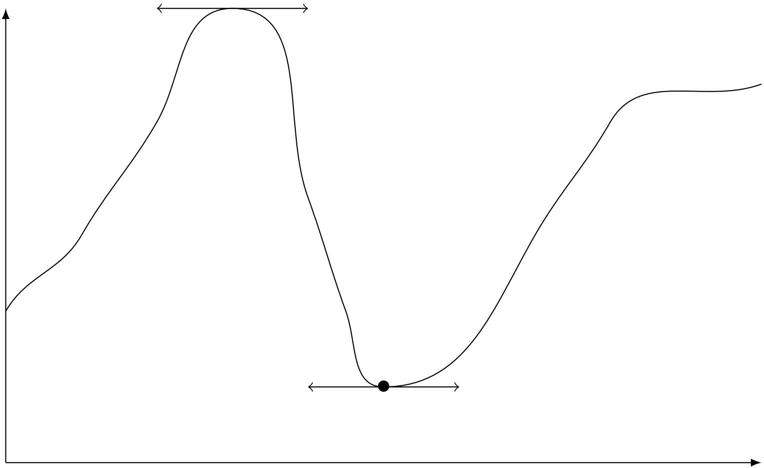
Unconstrained optimization



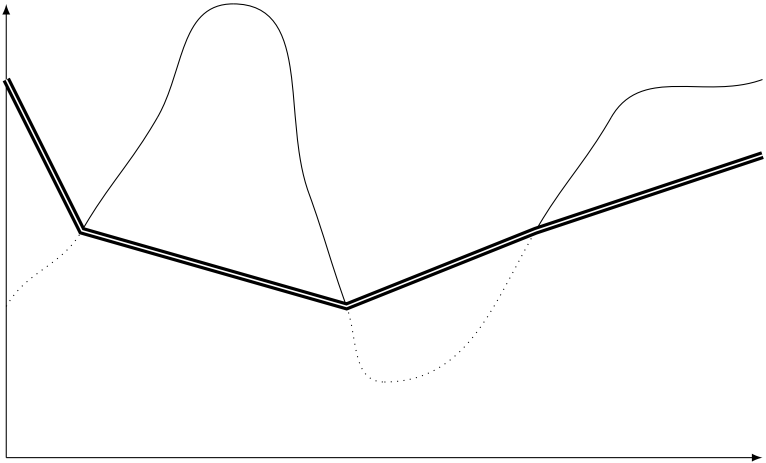
Unconstrainted optimization

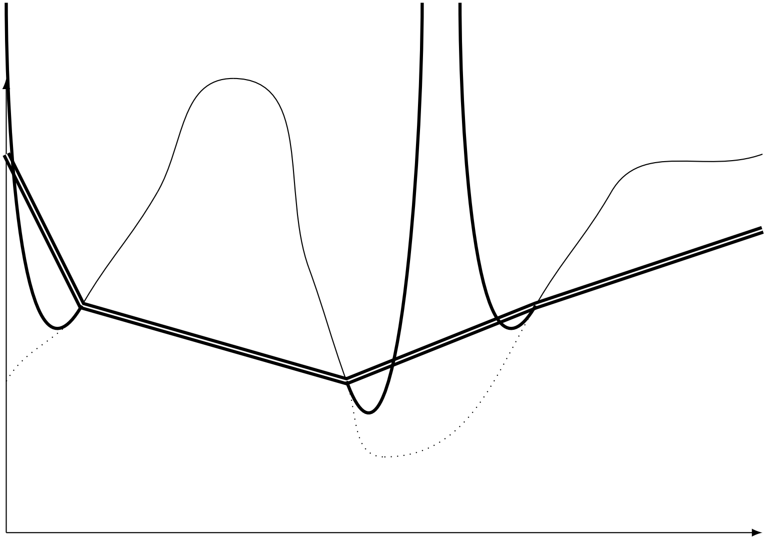


Unconstrained optimization

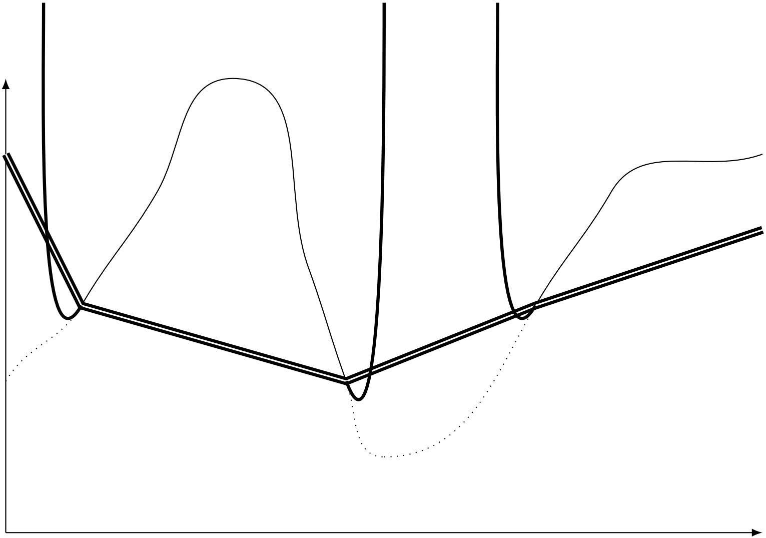


Constrainted optimization

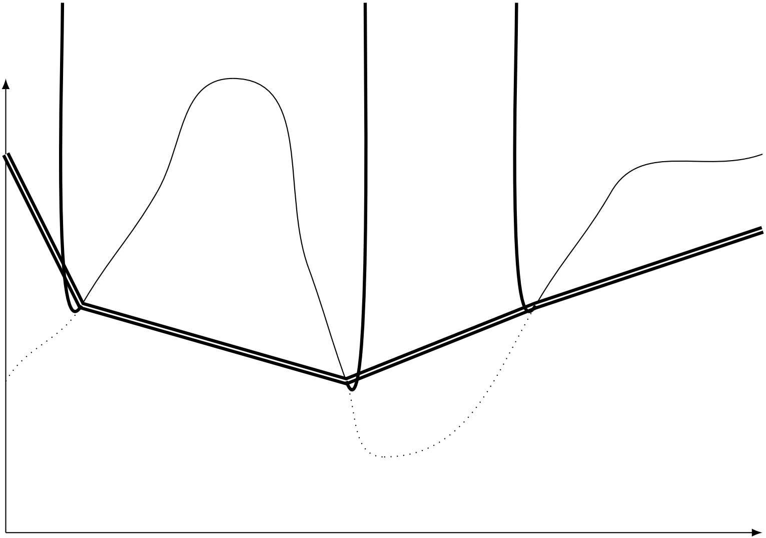




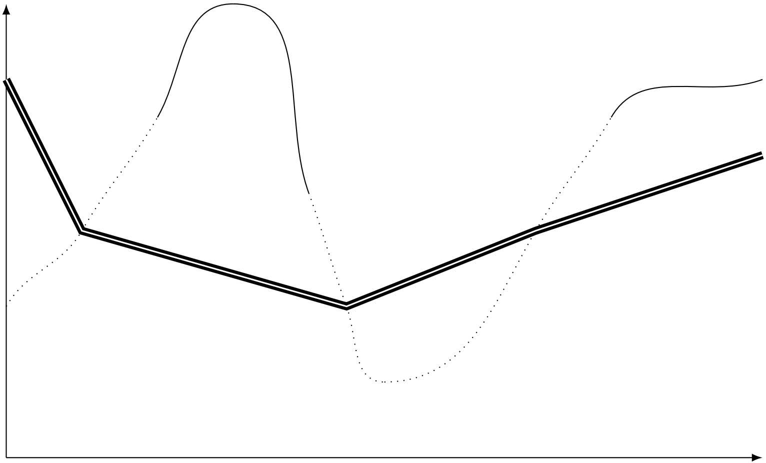
Penalty method

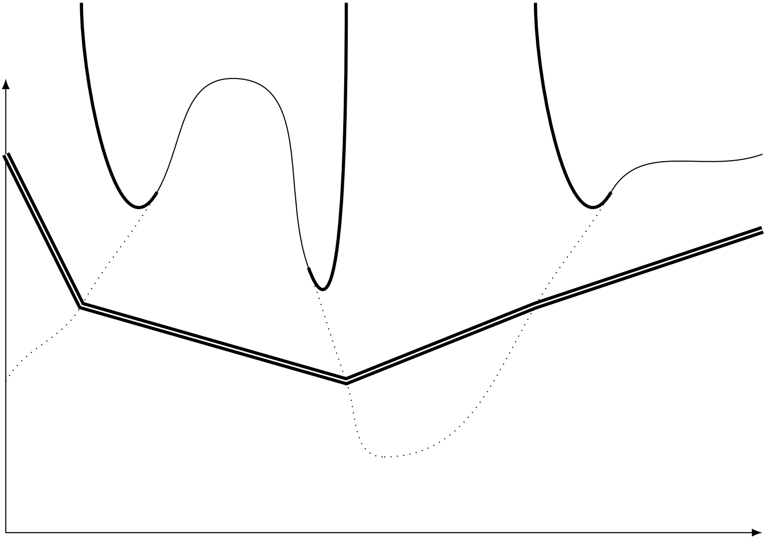


Penalty method

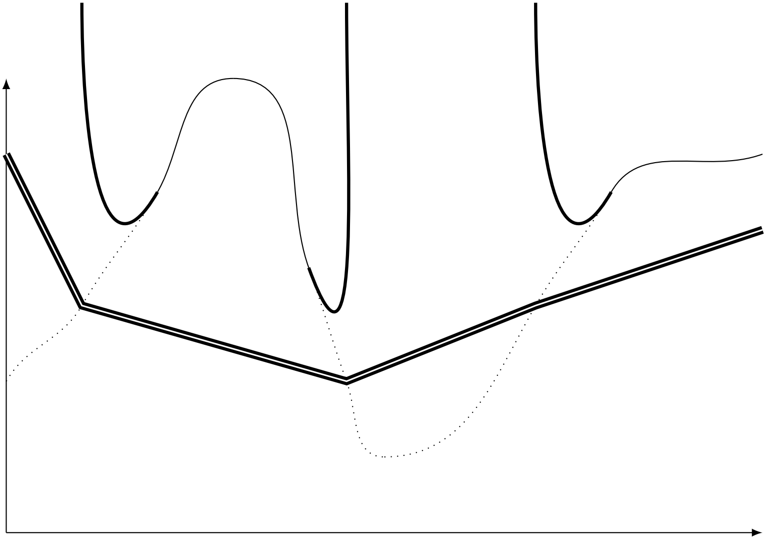


Barrier method

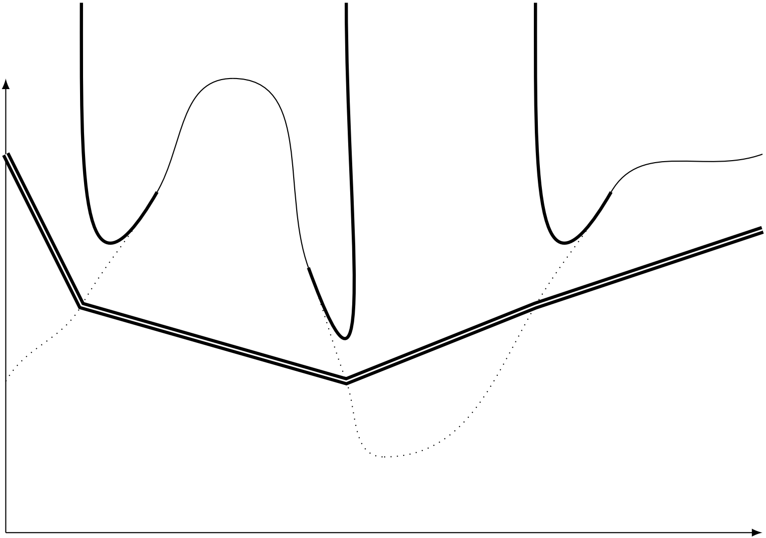




Barrier method



Barrier method



Optimization problem

We want to solve

$$\begin{array}{lcr} \min & f(x) & x \in \mathbb{R}^n \\ s.c. & g_i(x) \leq 0 & \forall\ 1 \leq i \leq m \\ & h_j(x) = 0 & \forall\ 1 \leq j \leq p \end{array}$$

We set

$$S = \{x | g_i(x) \leq 0\ \forall\ 1 \leq i \leq m; h_j(x) = 0\ \forall\ 1 \leq j \leq p\}$$

We want then to solve

$$\min_{x \in S} f(x)$$

We want

$$\min_{x \in S} f(x) \text{ et } x^* = \arg\min_{x \in S} f(x)$$

Penalties

Let $P : \mathbb{R}^n \rightarrow \mathbb{R}$

- P continuous
- $P(x) \geq 0$
- $P(x) = 0 \Leftrightarrow x \in S$

Solve $\lim_{\mu \rightarrow +\infty} \min_{x \in \mathbb{R}^n} f(x) + \mu P(x)$

Barrier

Let $B : \mathring{S} \rightarrow \mathbb{R}$

- B continuous
- $B(x) \geq 0$
- $B(x) \rightarrow +\infty \Leftrightarrow x$ get close to $Fr(S)$.

Solve $\lim_{\mu \rightarrow 0} \min_{x \in \mathring{S}} f(x) + \mu B(x)$

Constraint : $\mathring{S}$ not empty every neighborhood of $x^* \in S$ intersects $\mathring{S}$

Example of penalties and barriers

Courant-Beltrami Penalty

$$P(x) = \sum_{i=1}^{m} \max(0, g_i(x))^2 + \sum_{j=1}^{p} h_j(x)^2$$

Logarithmic barrier

If $-1 < g_i(x) < 0$ iff $x \in \mathring{S}$, $B(x) = \sum_{i=1}^{m} -\log(-g_i(x))$

Inverse barrier

If $g_i(x) < 0$ iff $x \in \mathring{S}$, $B(x) = \sum_{i=1}^{m} \frac{-1}{g_i(x)}$

Example, penalties method

We want to solve

$$\begin{array}{lll} \min & -x_1^2 - x_2^2 & x \in \mathbb{R}^2 \\ s.c. & x_1 + x_2 = C & \\ & -x_1 + \frac{1}{4} \leq 0 & \\ & -x_2 + \frac{1}{4} \leq 0 & \end{array}$$

Penalty :

$$P(x_1, x_2) = (x_1 + x_2 - C)^2 + \max(\frac{1}{4} - x_1, 0)^2 + \max(\frac{1}{4} - x_2, 0)^2$$

Example, barrier method

We want to solve

$$\begin{array}{lcr} \min & x_1^2 + x_2^2 & x \in \mathbb{R}^2 \\ s.c. & -x_1 - 2x_2 \leq -1 & \end{array}$$

Barrier :

$$B(x_1, x_2) = -\log(x_1 + 2x_2 - 1)$$

Convergence of the penalty method

We set $q(x, \mu) = f(x) + \mu P(x)$. We assume that f is continuous and admits at least one optimal solution. In addition, for every $\mu > 0$, $x_\mu = \arg\min_{x \in \mathbb{R}^n} q(x, \mu)$ exists.

Theorem

Let $(\mu_p)_{p \in \mathbb{N}}$ be a strictly increasing sequence. If $(x_{\mu_p})_{p \in \mathbb{N}}$ is convergent then it converges toward an optimal solution.

- $q(x_{\mu_p}, \mu_p) \leq q(x_{\mu_{p+1}}, \mu_{p+1})$
- $P(x_{\mu_p}) \geq P(x_{\mu_{p+1}})$
- $f(x_{\mu_p}) \leq f(x_{\mu_{p+1}})$
- $f(x_{\mu_p}) \leq q(x_{\mu_p}, \mu_p) \leq \min f(x)$

Theorem

If f is coercive then $\exists$ a subsequence of $(x_p)_{p \in \mathbb{N}}$ which is convergent.

Convergence of the barrier method

We assume that f is continuous and admits at least one optimal solution. In addition, for every $\mu > 0$, $x_\mu = \arg\min_{x \in \mathring{S}} f(x) + \mu B(x)$ exists.

Theorem

Let $(\mu_p)_{p\in\mathbb{N}}$ be a strictly decreasing sequence toward 0. If $(x_{\mu_p})_{p\in\mathbb{N}}$ is convergent then it converges toward an optimal solution.

- $q(x_{\mu_p}, \mu_p) \geq q(x_{\mu_{p+1}}, \mu_{p+1})$
- $B(x_{\mu_p}) \leq B(x_{\mu_{p+1}})$
- $f(x_{\mu_p}) \geq f(x_{\mu_{p+1}})$
- $\min f(x) \leq f(x_{\mu_p}) \leq q(x_{\mu_p}, \mu_p)$

Theorem

If f is coercive then $\exists$ a subsequence of $(x_{1/p})_{p\in\mathbb{N}}$ which is convergent.