Chapter 7 : Markov Chains

ENSIIE - Operations Research Module

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Random variables and process

Definition

A random variable X is a S-valued function where S is a set of **states**.

For instance, a dice roll has a value $X \in S = \{1, 2, 3, 4, 5, 6\}$.

Definition

A stochastic process is a family $(X_t)_{t \in T}$ of random variables where $T \subset \mathbb{R}^+$.

T is called the *time*, X_t is the state of X at time t.
For instance, we do a dice roll every second, X_t is the face of the dice after t seconds.

Markov process and time-homogeneous process

Definition

We say a process $(X_t)_{t \in T}$ is a *markov* process if and only if *the futur depends only on the present* :
$\forall t_1 < t_2 < \cdots < t_n < t_{n+1} \in T$, $\forall A \subset S$

$$P(X_{t_{n+1}} \in A|\ X_{t_1} X_{t_2}, \ldots, X_{t_n}) = P(X_{t_{n+1}} \in A|\ X_{t_n})$$

Definition

We say a process $(X_t)_{t \in T}$ is a *time-homogeneous* process if and only if *the conditional probabilities does not change when the time grows* :
$\forall t, t' \in T, s > 0 \backslash t + s, t' + s \in T, A \subset S$

$$P(X_{t+s} \in A | X_t) = P(X_{t'+s} \in A | X_{t'})$$

Markov chain

Definition

A *Markov chain* is a time-homogeneous process and a Markov process $(X_t)_{t \in T}$ where T is a countable set.

In addition, we assume S to be finite and discrete.

$T = \mathbb{N}$
Process: $X_1, X_2, \ldots, X_t, \ldots$
States: $1, 2, \ldots, |S|$

Transition probability

Definition

p_{ij} is the probability for the system to move from the state i to the state j in one step. This probabilty does not depend on the moment $t \in T$ when the state of the system is i.

$$\forall i, j \in S, \exists p_{ij} \quad \backslash \quad \forall t \in \mathbb{N} \quad P(X_{t+1} = j | X_t = i) = p_{ij}$$

Transition matrix:

$$P = \begin{array}{c} \begin{matrix} 1 & 2 & & j & & |S| \end{matrix} \\ \begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1j} & \cdots & p_{1|S|} \\ p_{21} & p_{22} & \cdots & p_{2j} & \cdots & p_{2|S|} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ p_{i1} & p_{i2} & \cdots & p_{ij} & \cdots & p_{i|S|} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ p_{|S|1} & p_{|S|2} & \cdots & p_{|S|j} & \cdots & p_{|S||S|} \end{pmatrix} \end{array} \begin{matrix} 1 \\ 2 \\ \\ i \\ \\ |S| \end{matrix}$$

Property

The sum of the elements of a row of P is always 1.

$$\sum_{j=1}^{|S|} p_{ij} = 1$$

Property : p-transitions

$$(P^2)_{ij} = P(X_{t+2} = j |\ X_t = i)$$
$$(P^3)_{ij} = P(X_{t+3} = j |\ X_t = i)$$

$$(P^p)_{ij} = P(X_{t+p} = j |\ X_t = i)$$

The states probability vector

Definition

We call $Q(t) = (q_1(t), q_2(t), \ldots, q_{|S|}(t))$ the *states probability vector*. $q_i(t)$ is the probability $P(X_t = i)$.

$$\sum_{i=1}^{|S|} q_i(t) = 1$$

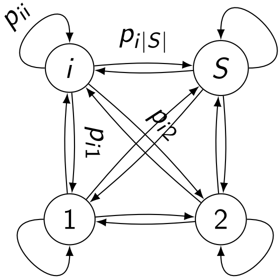
$$Q(t) = Q(t-1) \cdot P$$
$$Q(t) = Q(t-2) \cdot P^2$$
$$Q(t) = Q(0) \cdot P^t$$

Graph of a Markov chain

Definition

The graph $G = (V, A)$ of a Markov chain is a **directed graph** where every node is a state S (thus $V = S$), and every arc linking i to j is weighted with p_{ij}. An arc is not added is $p_{ij} = 0$.

Remark: the graph may contain loop arcs.



Remark

$(P^p)_{ij} > 0$ if there exists a path with p arcs between i and j.

State classification

Accessible states

A state j is *accessible* from i if there is a **path** from i to j in G.

$$\exists p \backslash (P^p)_{ij} > 0$$

Communicating states

Two states i and j are said *communicating* if i is accessible from j and conversely.

Communicating class

A *communicating class* is a strongly connected component of G. In other words, it is a maximal set of pairwise communicating states.

Irreducible chain

A chain with exactly one communicating class is *irreducible*.

Transient and recurrent states

Definition

A state i is *transient* if there is a state j such that j is accessible from i but i is not accessible from j.

It is possible to leave that state and never be able to come back.

Definition

A state i is *recurrent* if for every accessible state j from i, i is accessible from j.

It is always possible to visit again a permanent state.

Definition

A state i is *absorbing* if $p_{ii} = 1$.

Property

Theorem

In a communicating class, every state is reccurent or every state is transient.

Periodic states

$(P^p)_{ii} > 0$ if there exists a circuit with p arcs of weight non zero going through i.
Let $d_i = GCD(n \quad \backslash \quad (P^n)_{ii} > 0, n > 0)$.
(With $d_i = 1$ if $(P^n)_{ii} = 0 \quad \forall n$.)

Definition

A state i is said to be *periodic* if $d_i > 1$. Otherwise, the state is aperiodic. d_i is the *period* of i.

Property

An absorbing state is aperiodic.

Theorem

The period of every states in a communicating class is the same.

Convergence to a stationary distribution

Definition

A chain is said to be *regular* if the stationary distribution $Q^* = \lim_{t \to +\infty} Q(t)$ exists and is the same whatever $Q(0)$ is.

Theorem

A chain is regular if and only if every recurrent state is in the same class and if all those states are aperiodic.

Theorem

In a regular chain, $P^* = \lim_{p \to +\infty} P^p$ exists and every line of P^* is Q^*.

(proof on board)

Compute the vector Q^*.

Either, we use the following system :

$$Q(t+1) = Q(t) \cdot P \Rightarrow Q^* = Q^* \cdot P$$

AND

$$\sum_{i=1}^{|S|} q_i^* = 1$$

Or we compute P^*.