Chapter 8 : Queuing

ENSIIE - Operations Research Module

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2022

Poisson process

- States $S = \mathbb{N}$
- Time $T = \mathbb{R}^+$

Each time, an event can appear. $X(t)$ is the number of events that appeared between times 0 and t.

Poisson process

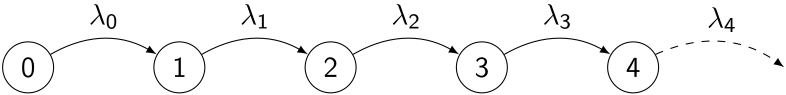
The probability an event appears between times t and $t + dt$ is $\lambda \cdot dt + o(dt)$.

$$Pr(X(t+dt) - X(t) = 1 | X(t) = n) = \lambda_n \cdot dt + o(dt)$$

$$Pr(X(t+dt) - X(t) = 0 | X(t) = n) = 1 - \lambda_n \cdot dt + o(dt)$$

$$(Pr(X(t+dt) - X(t) > 1 | X(t) = n) = o(dt))$$

Graphical representation



λ_n is the *arrival rate* : number of arrivals per unit of time when $X(t) = n$.

Probability $P_n(t)$

Let $P_n(t)$ be the probability that n events appear between times 0 and t.

Theorem

- $P'_n(t) = \lambda_{n-1} P_{n-1}(t) - \lambda_n P_n(t)$
- $P_0(0) = 1$
- $P_n(0) = 0$ si $n > 0$
- If $\forall n,\ \lambda_n = \lambda,\ P_n(t) = \frac{(\lambda t)^n \cdot e^{-\lambda t}}{n!}$

(Proof on board)

Queue

Definition

A queue merges two Poisson processes : a process where the events appear (birth process) and a process where the events disappear (death process).

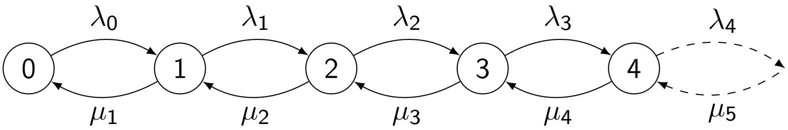
$$Pr(X(t+dt) - X(t) = 1 | X(t) = n) = \lambda_n \cdot dt + o(dt)$$

$$Pr(X(t+dt) - X(t) = -1 | X(t) = n) = \mu_n \cdot dt + o(dt)$$

$$Pr(X(t+dt) - X(t) = 0 | X(t) = n) = 1 - (\lambda_n + \mu_n) \cdot dt + o(dt)$$

$$(|Pr(X(t+dt) - X(t)| > 1 | X(t) = n) = o(dt))$$

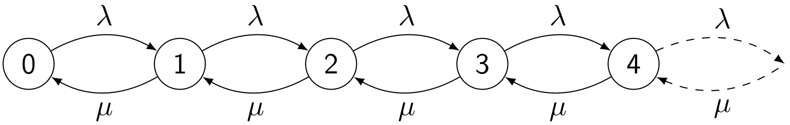
Graphical representation



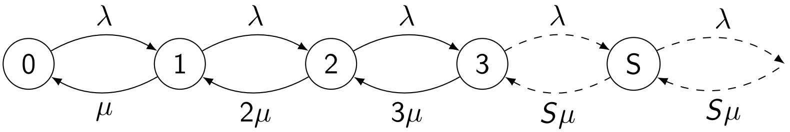
λ_n is the *birth rate* : number of arrivals every unit of time when $X(t) = n$.

μ_n is the *death rate* : number of departures per unit of time when $X(t) = n$.

Constant arrival, constant departure, 1 checkout.



Constant arrival, constant departure, S checkouts.



Each new checkout increases the death rate by μ.

Probability $P_n(t)$

Let $P_n(t)$ be the probability that $X(t) = n$.

Theorem

- $P'_n(t) = \lambda_{n-1} \cdot P_{n-1}(t) + \mu_{n+1} \cdot P_{n+1}(t) - (\mu_n + \lambda_n) \cdot P_n(t)$
- $P_0(0) = 1$
- $P_n(0) = 0, n > 0$

Stationary distribution

A stationnary distribution, if it exists, appears when $P_n(t)$ does not depend anymore on t :

- $\forall n, t \quad P_n(t) = P_n$
- $\forall n, t \quad P'_n(t) = 0$

Theorem

$P_n = \frac{\lambda_0 \cdot \lambda_1 \cdot \lambda_2 \cdots \lambda_{n-1}}{\mu_1 \cdot \mu_2 \cdot \mu_3 \cdots \mu_n} \cdot P_0$

Use $\sum_{n=0}^{\infty} P_n = 1$ to deduce P_0 and every other probability.

Proof on board

Necessary and sufficient condition

Theorem

The serie $\sum_{n=1}^{+\infty} \frac{\lambda_0 \cdot \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \cdot \mu_2 \cdots \mu_n}$ is convergent if and only if the stationary distribution exists.

Corollary

If $\lambda_n = \lambda$, $\mu_n = \mu$, and $\frac{\lambda}{\mu} < 1$ except for a finite number of values of n, then the stationary distribution exists.

Size of the queue

- Number of people in the queue (including the checkouts) :
$L = \sum_{n=0}^{+\infty} n \cdot P_n(t)$
- Number of people in the queue, not going through a checkout, if there are m checkouts : $L' = \sum_{n=m}^{+\infty} (n - m) \cdot P_n(t)$

Constant arrival, constant departure, 1 checkout.

We assume $\lambda_n = \lambda$ and $\mu_n = \mu$

Theorem

- Si $\frac{\lambda}{\mu} < 1$, then a stationnary distribution exists.
- $P_0 = (1 - \frac{\lambda}{\mu})$ and $P_n = (1 - \frac{\lambda}{\mu})(\frac{\lambda}{\mu})^n$
- $L = \frac{\lambda}{\mu - \lambda}$
- $L' = \frac{\lambda^2}{\mu(\mu - \lambda)}$

(Proofs on board)

Constant arrival, constant departure, S checkouts.

We assume $\lambda_n = \lambda$ and $\mu_n = \min(n, S)\mu$

Theorem

- Si $\frac{\lambda}{S\mu} < 1$, then a stationnary distribution exists.
- $P_0 = \frac{1}{\sum_{n=0}^{S-1} \frac{(\lambda/\mu)^n}{n!} + \frac{(\lambda/\mu)^S}{S!\left(1 - \frac{\lambda}{S\mu}\right)}}$ and $P_n = \left(\frac{\lambda}{\mu}\right)^n \frac{1}{S!} \frac{1}{S^{n-S}} P_0$
- $L = \left(\frac{1}{1 + \left(1 - \frac{\lambda}{S\mu}\right)\left(\frac{S!}{(\lambda/\mu)^S}\right) \sum_{n=0}^{S-1} \frac{(\lambda/\mu)^n}{n!}}\right) \frac{\frac{\lambda}{S\mu}}{1 - \frac{\lambda}{S\mu}} + \frac{\lambda}{\mu}$
- $L' = P_0 \cdot \left(\frac{\lambda}{\mu}\right)^S \frac{1}{S!} \frac{\lambda}{S\mu} \left(\frac{1}{1 - (\lambda/(S\mu))}\right)^2$

(Proofs on board)