Tutorial 6 : Projected Gradient algorithm

Operations research, 3rd semester.

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Exercice 1 — *A simple example.*

Let (P) be the following problem :

$$\min_{x\in\mathbb{R}^2} f(x) = x_1^2 + 4x_2^2 \text{ s.c. } \begin{cases} x_1 + 2x_2 \geq 1 \\ -x_1 + x_2 \leq 0 \\ x_1 \geq 0 \\ x_2 \geq 0 \end{cases}$$

1. Draw the graphical representation of the problem.
2. Check that every feasible solution x of (P) (i.e. satisfying the constraints) satisfies the linear independance qualification. Deduce that only the optimale solutions satisfies the Karush Kuhn Tucker conditions.
3. Apply the algorithm from $P_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ (3 iterations).
4. Check that the point satisfies the Karush Khun-Tucker conditions.

Exercice 2 — *With equalities*

Let (P) be the following problem :

$$\max_{x\in\mathbb{R}^2} f(x) = x_1 \cdot x_2 \text{ s.c. } \begin{cases} x_1 + x_2 + x_3 = 4 & (1) \\ x_1 - x_2 + x_4 = 1 & (2) \\ x_1, x_2, x_3, x_4 \geq 0 \end{cases}$$

Same questions as exercice 1 except that the algorithm should be started from the point $\begin{pmatrix} 1.5 \\ 0.5 \\ 2 \\ 0 \end{pmatrix}$).

Exercice 3 — *Projection operator.*

Given a vector $g \in \mathbb{R}^n$, we want to minimize the $\frac{1}{2}\|g - p\|^2$ where such that $p \in \mathbb{R}^n$ and $Ap = 0$. The size of the matrix A is $m \times n$ with rank $m < n$.

In other words, we search for the projection of g over the space $\{p \backslash Ap = 0\}$.

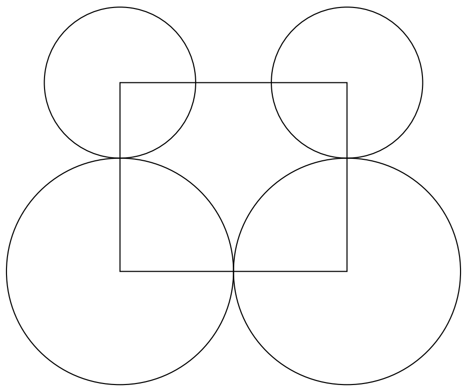
1. Write and solve the (KKT) conditions for this problem.
2. This way, find the formula of the projection operator on the space $L = \{p \backslash Ap = 0\}$.

Exercice 4 — *Non convex problem*

We consider the following problem :

Given a rectangle of size 10×12, we want to place 4 disks, centered in each of the 4 vertices of the rectangle such that the area is **maximum** and such that the interiors of the disks do not intersect ; in other words, the disks may only touch the boundaries of the others.

For example, the following drawing contains a maximal feasible solution :



1. Modelize the problem as a mathematical program. We set $\gamma = \sqrt{244} \simeq 15.62$.
2. Show that, if we apply the projected gradient algorithm from the point where every radius is nul, the algorithm stops at the first iteration.
3. Show that, if we apply the projected gradient algorithm from the point given on the drawing (we may assume that the radius of the two lower disks are equals), the algorithm stops at the first iteration.
4. Show that those solutions are not optimal.

Exercice 5 — *Projection and standard form*

We consider the following programs :

$$\min_{x \in \mathbb{R}^2} f(x) = x_1^2 + x_2^2 \text{ s.c. } \begin{cases} x_1 + x_2 \geq 2 \\ x_1 \geq 0 \\ x_2 \geq 0 \end{cases}$$

$$\min_{x \in \mathbb{R}^2} f(x) = x_1^2 + x_2^2 \text{ s.c. } \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 \geq 0 \\ x_2 \geq 0 \\ x_3 \geq 0 \end{cases}$$

1. Show that the two programs are equivalent. (They have the same optimal value and every optimal solution of hte first program can be deduced from an optimal solution of the other.)
2. Draw the graphical representation of the two programs on the same drawing.
3. Show that if we apply an iteration of the projected gradient algorithm on the first program from the point $x = (3, 2)$, the direction we follow is $(-6, -4)$.
4. Show that if we apply an iteration of the projected gradient algorithm on the second program from the point $x = (3, 2, 3)$, the direction we follow is $(-8/3, -2/3, -10/3)$.
5. How can we explain that two equivalent programs do not follow the same direction on the drawing ?