Tutorial 7 : Reduced Gradient algorithm

Operations research, 3rd semester.

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Exercice 1 — ***Simple example.***

Soit (P) le problème suivant :

$$\min_{x\in\mathbb{R}^2} f(x) = x_1^2 + 4x_2^2 \;\text{ s.c. } \begin{cases} x_1 + 2x_2 \geq 1 \\ -x_1 + x_2 \leq 0 \\ x_1 \geq 0 \\ x_2 \geq 0 \end{cases}$$

1. Write the augmented form of (P) using two new variables x_3 and x_4.
2. Draw the graphical representation of the problem, include x_3 and x_4 on the drawing.
3. Apply the algorithm from $P_0 = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 0 \end{pmatrix}$ and the basis $\mathcal{B} = \{1, 3\}$ (3 iterations).
4. Check that the point satisfies the Karush-Khun-Tucker conditions.

Exercice 2 — ***Reduced gradient and standard form***

Let (P) be the following program.

$$\min_{x\in\mathbb{R}^n} f(x) \;\text{ s.c. } \begin{cases} \displaystyle\sum_{j=1}^{n} a_{ij}x_j \leq b_i \;\forall i \in [\![1; m]\!] \\ x \geq 0 \end{cases}$$

We assume that the functions g_i are linear and that there exists a feasible solution $z > 0$.

1. Write the augmented form of (P) by adding m variables. Let $y_1, y_2, \ldots, y_{n+m}$ be the variables of that new program and f' be the new objective function.
2. Show that if we set $\begin{pmatrix} y_1 \\ y_2 \\ \cdots \\ y_n \end{pmatrix}$ to z then, for every $i \in [\![1; m]\!]$, $y_{n+i} \geq 0$. Let y_0 be that solution.
3. Show that it is possible to start the reduced gradient algorithm from y_0 with the basis $B = \{n + i, i \in [\![1; m]\!]\}$.
4. Compute the direction d obtained at the first iteration and show that $i \leq n$, $d_i = -(\nabla f(z))_i$.

Exercice 3 — *Linear objective*

Let (P) be the following program :

$$\min_{x \in \mathbb{R}^n} c \cdot x \quad \text{s.c.} \quad \begin{cases} Ax = b \\ \quad x \geq 0 \end{cases}$$

where A is a $m \times n$ matrix with $m \leq n$ and where b is a vector of size m.

We can solve such a program with the simplex algorithm, which is similar to the reduced gradient algorithm. We recall that each non-basic variable is nul.

1. Let x be a basic feasible solution : a feasible solution and a basis B such that for all $i \in N$, $x_N = 0$. Compute the reduced cost gradient and, this way, find the formula of the reduced costs of the simplex algorithm.
2. We assume that $Ax = b; x \geq 0$ is a bounded space. Show that, during an iteration of the reduced gradient algorithm, if $d \neq \vec{0}$, there is necessarily a variable of x that is nul at the end of the iteration.
3. Is there necessarily a change of basis in that case ?
4. We recall that the simplex algorithm moves, at each iteration, by interverting two variables from the basis and the non-basis. Show that, even if we start at the same point with the same basis, there exists cases where the reduced gradient algorithm and the simplex algorithm choose different directions.