Tutorial 8 : Penalties and Barrier methods

Operations research, 3rd semester.

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Exercice 1 — ***Beltrami penalties***

We want to minimize $f(x) = x_1 + x_2$ such that $x_1^2 - x_2 \leq 2$.

1. Solve the problem using the penalties method, with the penalty of Courant-Beltrami.
2. Same question with the equivalent following problem : minimize $f(x) = x_1 + x_2$ such that $x_1^2 - x_2 + x_3^2 = 2$.

Exercice 2 — ***Barrier method***

Solve the following problem with the barrier method : minimize $f(x) = x_1 + x_2$ such that $x_1^2 - x_2 \leq 2$.

Exercice 3 — ***Penalties and Lagrange multipliers***

We consider a generic problem in which we want to minimize $f(x)$ on $\mathbb{R}^n$ such that $g_i(x) \leq 0$ for every $i \in [\![1; m]\!]$ and such that $h_j(x) = 0$ for every $j \in [\![1; p]\!]$. We apply the penalties method with the Penalty of Beltrami P. We write $q(x, \mu) = f(x) + \mu P(x)$ and $x_k = \arg\min q(x, k)$.

We assume that f, g_i and h_j are C^1. Let x^* be an optimal solution of f, we assume x^* satisfy the linear independant constraint of qualification. Finally, we assume that the sequence x_k converges toward x^*.

1. Recall the Kuhn-Tucker conditions for that problem at x^*, we write λ_i and μ_j the Langrange multipliers respectively associated with the functions g_i and h_j.
2. May there exist multiple values for λ_i, μ_j satisfying the conditions ?
3. Write the gradient of $q(x, k)$ at x_k, what is the numerical value of that gradient ?
4. Deduce that $\lim\limits_{k \to +\infty} 2k g_i^+(x_k) = \lambda_i$ and $\lim\limits_{k \to +\infty} 2k h_j(x_k) = \mu_j$ if those limits exist.

Exercice 4 — ***Internal barrier***

We consider the following problem

$$\min_{x \in \mathbb{R}} f(x) = x^2 \quad \text{s.c.} \begin{cases} x - 1 \leq 0 & (1) \\ -x - 1 \leq 0 & (2) \\ -x^2 \leq 0 & (3) \end{cases}$$

1. Draw the graphical representation of the problem and an associated logarithmical barrier. Does the problem satisfy the hypothesis that are needed to use the logarithmical and inverse barriers ?
2. Whatever the answer to the previous quesiton is, try to solve the problem using a barrier method, using a logarithmical barrier.
3. We replace the constraint $-x^2 \leq 0$ by $g_3(x) \leq 0$ with

$$g_3(x) = \quad \text{s.c.} \begin{cases} -(x + \frac{1}{2})^2 \text{ if } x \leq \frac{-1}{2} \\ -(x - \frac{1}{2})^2 \text{ if } x \geq \frac{1}{2} \\ 0 \text{ otherwise} \end{cases}$$

   Draw the graphical representation of the problem and an associated barrier.
4. Can we solve the problem with the barrier method ? How could we get around that problem ?