Tutorial 7 : Markov chains

Operations research, 3rd semester.

2024

Exercice 1 — *Rock Paper Scissors Lizard Spock*

The rules of Rock Paper Scissors adapted by Dr Sheldon Cooper are the following : scissors cuts paper, paper covers rock, rock crushes lizard, lizard poisons Spock, Spock smashes scissors, scissors decapitates lizard, lizard eats paper, paper disproves Spock, Spock vaporizes rock and, as it always has, rock crushes scissors.

A tournament is about to start. A player wins a versus battle if he is the first to win 100 rounds. One of the players, Leonard, studied one of his opponents, Howard. He deduced the following rules :

- when Hower plays Rock, he then plays uniformly one of the other shapes ;
- when he plays Lizard, he then plays Lizard again ;
- when he plays Paper, he then plays Scissors ;
- when he plays Spock, he then plays Lizard or Spock, he plays Lizard four times more than Spock ;
- finally, when he plays Scissors, he then plays Rock, Spock or Lizard, he plays Spock twice more than Lizard and he plays Rock twice more than Lizard too.

We can modelize the way Howard plays by a Markov chain.

1. What is the stochastic process $\{X_t \in S\}_{t \in T}$ of that chain : what are the states S, the time T and why is this process a Markov chain ?
2. What is the transition matrix and the associated graph ?
3. What are the communicating class ? Is the chain irreducible ?
4. Which states are transient, recurrent or absorbing ?
5. What is the probability that Howard plays Rock 3 rounds after playing Rock ? Lizard 3 rounds after playing Spock ?
6. What are the 2-transitions probabilities ? (the probability of playing i 2 rounds after j for all i and j).
7. We assume Howard plays his first turn uniformly, after how many turn Howard has a 1 in 2 chance playing Lizard ? 8 in 10 chances playing Lizard ?
8. How Leonard must choose his next shape in order to maximize his chances of winning ? to minimize his chance of loosing ?
9. Is the chain regular ? If so, what is the stationary distribution ? Deduce how Leonard must play in order to win against Howard.

Exercice 2 — *Advertising*

There are three competiting products P_1, P_2 and P_3. We know that 30% of surveyed people prefer P_1, 350% prefer P_2 and the rest prefer P_3. Using an advertising campaign, the society selling P_1 tries to increase its market share. After the campaign, we analyze which clients changed their minds :

before \ after	P_1	P_2	P_3
P_1	50%	40%	10%
P_2	30%	70%	0%
P_3	20%	0%	80%

For example, we see that 20% of the people who prefered P_2 now prefer P_1.

We can modelize the campaign effect by a Markov chain.

1. What is the stochastic process $\{X_t \in S\}_{t \in T}$ of that chain : what are the states S, the time T and why is this process a Markov chain ?
2. What is the transition matrix and the associated graph ?
3. What are the market shares after the campaign ?
4. We redo the same campaign, we assume the effects are the same. Give, for each product P, how many percents of people of preferred P before the first campaign are now preferring P_1, P_2 or P_3.
5. What are the market shares after the second campaign ?
6. We assume the campaign is redone indefinitely. Does a market shares limit exist ? In that case, what is it ?

Exercice 3 — ***Work policy***

A public work society has a team where every member works on the same work site. It can work on 2 kinds of work sites : medium works (1 week, type A), or long works (2 weeks, type B). Statistically, every Monday, the society receives with a 1 in 2 chance a request of type A, and with a 3 in 5 chances a request of type B. The requests are independent : the society can receive a request of type A and a request of type B the same week. In that case, the team always choose the type B request. The team cannot work on two sites at the same time : if it is working on a type B work site and receives a request, it is ignored.

In case of a type A work, the society receives 500 euros. It earns 1200 euros at the end of a type B work. Finally, it looses 250 euros every inactive week.

We can modelize the activity every week by a Markov chain.

1. What is the stochastic process $\{X_t \in S\}_{t \in T}$ of that chain : what are the states S, the time T and why is this process a Markov chain ?
2. What is the transition matrix and the associated graph ?
3. What are the communicating class ? Is the chain irreducible ?
4. Which states are transient, recurrent or absorbing ?
5. We assume the society works indefinitely. Does a stationary distribution exist ? In that case, what is the mean profit every week ?
6. Should the society choose the type A work instead of the type B work when the two requests simultaneously occurs ?

Exercice 4 — ***Energy consumption***

A familly uses their television this way : when they look at the television, more or less one hour after, they have 50% chances of continuing looking at it, 33% chances of putting it in sleep mode. The last case consists in shutting it down. After 2h of sleep mode, the TV automatically stops. Every hour, there is 10% chances that someone starts the TV. It consumes 28Wh per hour when it is switched on and 2.5Wh per hour when it is in sleep mode. What it the mean annual consumption in Wh of this television ?

Exercice 5 — ***Festivities***

A big event is organized. The place is cut in 5 zones :

- the entrance E where people come, pay and leave.
- the cloackroom V where people put their coats.
- Three main zones, the transporation zone T, the logistics zone L and the network zone R.

1. According to some preliminary studies, we know that every minut,
   - Every visitor leave the entrance for the cloackroom with a probability $\frac{1}{30}$. Otherwise they stay in the entrance.

- Every visitor leave the cloackroom with a probability $\frac{1}{15}$. In that case, they go to the main zones with the same probvability. Otherwise they stay in the cloackroom.
- A visitor in the zone T has one chance over two to stay, on chance over four to go in R and one chance over four to go in L.
- A visitor in the zone R has one chance over two to stay, on chance over four to go in T and one chance over four to go in L.
- A visitor in the zone L has one chance over two to stay, on chance over four to go in T and one chance over four to go in R.

Use a markov chain to model the movements of the people. Do the graphical representation of the chain.

2. What are the transient and recurrent states ?
3. We would like to add food stands in the zones T, R and L. Every stand increase by 0.1 the probability of staing in the zone, and reduces the probability to go to the each of the other zones by 0.05. We set A (respectively B and C) the number of stands in the zone T (respectively L and R). We want to study how the chain evolve as a function of A, B and C. In order to simplify the calculations, we set $\alpha = 0.1A$; $\beta = 0.1B$ and $\gamma = 0.1C$; then for instance α is exactly the increase of the probability of staying in T.

   Change the chain of question 1 to introduce the values α, β and γ.
4. Show that, if among α, β and γ, at least two values are equal to $\frac{1}{2}$, there is no stationary distribution
5. What is the stationary distribution if $\alpha = \frac{1}{2}$ and $\beta, \gamma \neq \frac{1}{2}$.
6. In the case where $\alpha, \beta, \gamma \neq \frac{1}{2}$, show that the stationary distribution as a function of α, β and γ is $(q_E^* = 0, q_V^* = 0, q_T^*, q_L^*, q_R^*)$ with

$$q_T^* = \frac{1-2\beta}{1-2\alpha} q_L^* \qquad q_L^* = \frac{1-2\alpha}{1-2\beta} q_T^* \qquad q_R^* = \frac{1-2\alpha}{1-2\gamma} q_T^*$$
$$q_T^* = \frac{1-2\gamma}{1-2\alpha} q_R^* \qquad q_L^* = \frac{1-2\gamma}{1-2\beta} q_R^* \qquad q_R^* = \frac{1-2\beta}{1-2\gamma} q_L^*$$

   Et en déduire que

$$q_T^* = \frac{1}{1 + \frac{1-2\alpha}{1-2\beta} + \frac{1-2\alpha}{1-2\gamma}}$$
$$q_L^* = \frac{1}{1 + \frac{1-2\beta}{1-2\alpha} + \frac{1-2\beta}{1-2\gamma}}$$
$$q_R^* = \frac{1}{1 + \frac{1-2\gamma}{1-2\beta} + \frac{1-2\gamma}{1-2\alpha}}$$

7. What should be the values of A, B and C is we want to have twice more people in T and L and R after some time ?