Tutorial 8 : Queuing

Operations research, 3rd semester.

2024

Exercice 1 — ***File d'attente (6 points)***

We consider a queue. For each of the following cases, where we describe the birth rate λ_n in arrival per second and the death rate μ_n in leaving per second par seconde if there exists a stationary distribution. In that case, what is the value P_n for every $n \geq 0$. We recall thatexp$(x) = \sum_{n=0}^{+\infty} \frac{x^n}{n!}$.

1. $\lambda_n = 3$, $\mu_n = 5$
2. $\lambda_n = 5$, $\mu_n = 3$
3. $\lambda_n = (n+1)$, $\mu_n = 100$
4. $\lambda_n = (n+1)$, $\mu_n = n^2$

Exercice 2 — ***Supermarket queue***

In a supermarket, 5 clients can go through a checkout every 10 minutes. While there are 2 clients or less in the queue, 2 clients come every 5 minutes. After that, 15 clients appear every 20 minutes. If there are 9 clients or less, only one checkout is open. If there are more than 10 clients, 2 more checkouts open.

1. What is the graphical representation of that queue ?
2. What is the value of $P'_n(t)$ for every n as a function of $P_m(t)$ for every $m \in \mathbb{N}$?
3. Prove that a stationary distribution exists.
4. What is the value of P_n for every n in the stationary distribution ?
5. What is the mean number of people waiting in the queue (and thus not going through a checkout) ?

Exercice 3 — ***Interversion of the birth and death rates***

We consider a paradise (for instance, of the pastafarian religion). A dead person comes to the paradise until it is reincarnated and rebirth on Earth.

The mean number of death is 2000 people per second in the world. Those numbers do not depend on the number of people in the paradise.

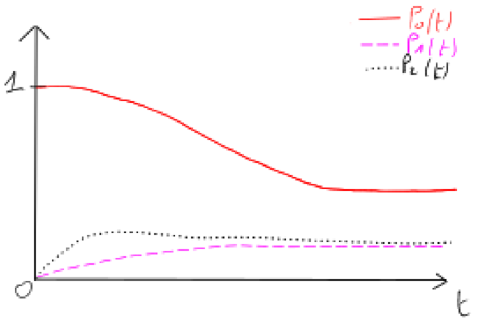
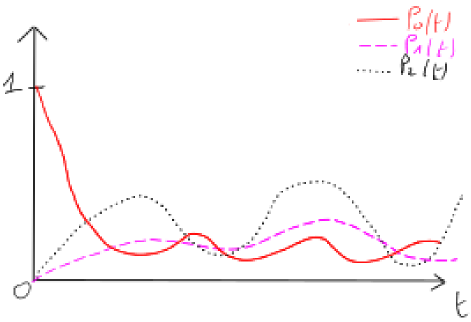
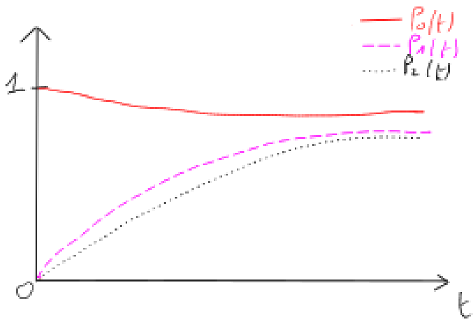
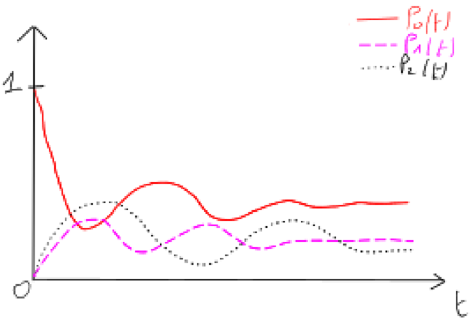
At the reincarnation service, two pirates are in charge : Blackbeard and Barbarossa. Blackbeard reincarnates 60000 people per minutes on average and works only if there are 10000 people or less in the paradise. Barbarossa reincarnates 180000 people on average and works only if there are 5001 people or more in the paradise.

Let $\lambda = 2000$, $\mu = 1000$, $\mu' = 3000$, $A = 5000$ and $B = 10000$.

1. Draw the graphical representation of the queue process associated with the deaths and reincarnations in the Pastafarian paradise. Use the notations λ, μ, μ', A et B.
2. If we assume that there exists a stationary distribution, why can we assume that this distribution is reached ?
3. Show that the stationnary distribution exists.
4. Let P_n be the probability that there are n people in the paradise. Describe, as a function of λ, μ, μ', A, B, P_0 and n, the probability P_n. Using the numerical values, obtain P_n as a function of A, P_0 and n.
5. Show that $P_0 = 1/\left(2^{A+1} + 2^A\right)$.

Exercice 4 — *Queues and curves* We consider a queue with a birth rate λ_n and a death rate μ_n. We assume that after some value n, the rates are constant and equal μ et λ, and we assume that $\mu > \lambda$.

We drawn 4 graphics with three curves $P_0(t)$, $P_1(t)$ and $P_2(t)$ depending on t. For each graphic, could the curves correspond to the probability that there are 0, 1 or 2 people in the queue ?

Exercice 5 — *Queue of pairs*

A famous hairdresser is in high demand but does not take appointments. So there is always a long queue. We want to measure it. We realize that there are in fact people going to this hairdresser because he offers discount vouchers for pairs of people. So we must consider that people arrive not one by one in the queue but two by two. On the other hand, they always leave one by one. We therefore rewrite the arrival and departure probabilities as follows :

$$\begin{aligned}
&Pr(X(t+dt)-X(t)=2 & |X(t)=n) = && L_n dt + o(dt)\\
&Pr(X(t+dt)-X(t)=-1 & |X(t)=n>0) = && \mu_n dt + o(dt)\\
&Pr(X(t+dt)-X(t)=0 & |X(t)=n>0) = && (1-L_n-\mu_n)dt + o(dt)\\
&Pr(X(t+dt)-X(t)=0 & |X(t)=0) = && (1-L_0)dt + o(dt)\\
&Pr(X(t+dt)-X(t)\notin\{-1,0,2\} & |X(t)=n) = && 0 + o(dt)
\end{aligned}$$

We write $P_n(t)$ the probability that there are n people in the queue at time t and P_n the probability in the stationary distribution if it exists.

1. Do the graphical representation of the queue.
2. Show that $P_n'(t) = P_{n-2}(t)\cdot L_{n-2} - P_n(t)\cdot(L_n+\mu_n) + P_{n+1}(t)\cdot\mu_{n+1}$ if $n > 2$.
3. What are the formulas of $P_0'(t)$ and $P_1'(t)$.

4. Assuming there exists a stationary distribution, show that $P_1 = \frac{L_0}{\mu_1} P_0$ and $P_2 = (L_1 + \mu_1) \cdot \frac{L_0}{\mu_1 \mu_2} P_0$.

We set for $n \geq 0$:

$$NS(n) = \{I \subseteq [\![1;n]\!] | \forall i < j \in I, i+1 \neq j\}$$
$$P(I,n) = \prod_{i \in I} \mu_i \prod_{\substack{i \notin I \\ i \leq n}} L_i$$

For instance $NS(5) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{1,3\}, \{1,4\}, \{1,5\}, \{2,4\}, \{2,5\}, \{3,5\}, \{1,3,5\}\}$ et $P(\{1,4\},4) = \mu_1 L_2 L_3 \mu_4$.

Note that : $NS(0) = \emptyset$, $P(\emptyset, 0) = 1$.

5. Show that, for $n \geq 1$, $\sum\limits_{\substack{I \in NS(n+1) \\ n+1 \in I}} P(I, n+1) = \mu_{n+1} \sum\limits_{\substack{I \in NS(n) \\ n \notin I}} P(I,n)$

   Clue : ***do not*** *use a proof by induction.*

6. Show that, for $n \geq 1$, $\sum\limits_{\substack{I \in NS(n+1) \\ n+1 \notin I}} P(I, n+1) = L_{n+1} \cdot \sum\limits_{I \in NS(n)} P(I,n)$

   Clue : again, no induction

7. Show that, for $n \geq 1$, $\sum\limits_{I \in NS(n+1)} P(I, n+1) = L_{n+1} \cdot \sum\limits_{I \in NS(n)} P(I,n) + \mu_{n+1} \cdot \sum\limits_{\substack{I \in NS(n) \\ n \notin I}} P(I,n)$

   Clue : and again, no induction

8. Show that, for $n \geq 1$, $P_n = \sum\limits_{I \in NS(n-1)} P(I, n-1) \cdot \frac{L_0}{\prod_{i=1}^{n} \mu_i} P_0$

   Clue : use the three previous questions. In this case you can use an induction formula.

9. 60 people arrive every hour and 35 leave every half hour. What is the value of P_8 ?