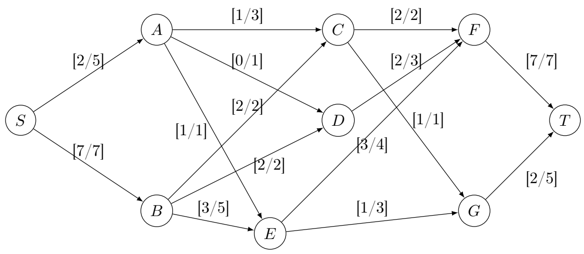
Tutorial 4 : The maximum flow and the minimum cut problems

Operations research, 3rd semester.

2024

Exercice 1 — *The Ford-Fulkerson algorithm*

1. Compute the maximum flow problem in the following graph with the Ford-Fulkerson algorithm. An initial flow is given. Use the marking algorithm. Do one of the iterations again with the residual network method.
2. Give a minimum cut of this graph.
3. Which capacity should we change in this graph in order to increase the maximum flow by 1 ? Give a counterexample in which it is not sufficient. Describe an algorithm that change the capacity of one or more arcs and increase the maximum flow by exactly 1.



▶ **Correction**

By solving question 1, one finds a flow of 11, for instance with the min cut (SABEDCF ; GT), which includes the arcs (FT, CG, EG) with capacities 7+1+3 = 11.

An augmenting path that can be found is $SACBEGT$ with the following labeling : A labeled with $+S$, C labeled with $+A$, D labeled with $+A$, B labeled with $-C$, E labeled with $+B$; F labeled with $+D$, G labeled with $+E$, and T labeled with $+G$. This path increases the flow by 2.

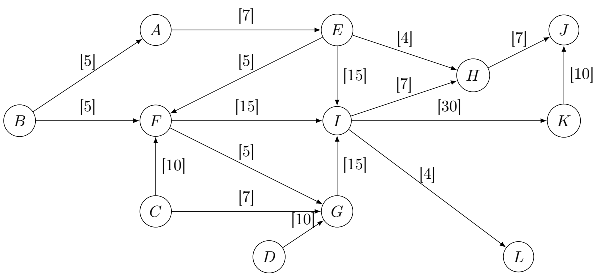
Another augmenting path that can be found is $SADFEGT$ with the following labeling : A labeled with $+S$, C labeled with $+A$, D labeled with $+A$, B labeled with $-C$, F labeled with $+D$, E labeled with $-F$; G labeled with $+E$, and T labeled with $+G$. This path increases the flow by 1.

For question 3, the expected answer is "The arcs of the min cut" and this works here. However, in a graph where there are multiple min cuts, this is not sufficient. A simple example is S -[1]-> A -[1]-> T.

where there are 2 min cuts (S, AT) and (SA, T) with a capacity of 1. The correct answer is therefore "increase the capacity of the arcs of all the min cuts." An algorithm to increase the max flow by 1 is as follows : - calculate a max flow of value v - as long as the max flow is of value v ; find a min cut, choose an arc from this min cut and increase its capacity by 1.

Exercice 2 — *Pipes of water*

Water is supplied to three cities J, K and L from four stocks A, B, C, D (water towers, groundwater table, ...). Each day, those stocks produce respectively 15, 10, 15 and 15 thousands of m^3 of water. The pipes of the city are represented on the following graph where the maximum flow rate in thousands of m^3 per day of each pipe is written next to the arc.



1. Compute how many m^3 of water can be piped to the cities. To do so, transform this problem into a maximum flow problem and compute a minimum cut. Use the Ford-Fulkerson algorithm with the marking algorithm. Do one of the iterations again with the residual network method.
2. This value does not seem sufficient. Those three cities want to improve the network so that it can meet more needs of the population. They made a study to determine how much water they need each day : 15000 m^3 for J, 20000 for K and 15000 for L. The town council decided to rebuild the pipes (A,E) and (I,L).

   What capacity should we associate to those two arcs in order to fulfil the needs of the cities.

▶ **Correction**

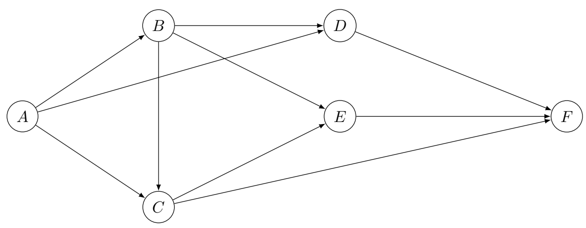
1. We add a source s and edges connecting it to A, B, C, and D with capacities 15, 10, 15, and 15. We add a sink t and edges connecting J, K, and L to t with infinite capacity.

   We find, for example, the following augmenting paths :
   - $sAEIHJt$: value + 7
   - $sBFIKt$: value + 5
   - $sCGIKt$: value + 7
   - $sDGIKt$: value + 8
   - $sCFILt$: value + 4
   - $sCFIKt$: value + 4
   - $sDGCFIKt$: value + 2

   Value 37.
2. To determine these new capacities, we can proceed as in exercise 1, looking for a minimum cut, increasing (A, E) or (I, L) if these pipelines are part of the cut. If neither of these edges is in a cut, then the objective cannot be reached.

Exercice 3 — *Disjoint paths in a graph*

Let G be the following graph.



What is the maximum number of node disjoint paths from A to F in this graph ? (Three paths are node disjoint if and only if its do not have any common node.) In order to answer this question, transform this problem into a maximum flow problem.

▶ **Correction**

To find arc-disjoint paths between A and F, we assign a capacity of 1 to all edges and look for a maximum flow. To find node-disjoint paths, we duplicate each node v into v^- and v^+ ; incoming arcs go to v^-, and outgoing arcs come from v^+. Finally, we add an arc from v^- to v^+ with a capacity of 1. We seek a maximum flow from A^+ to F^-.

Exercice 4 — *Maximum matching*

We recall that a bipartite graph is a graph where the nodes are parted into two independent sets (nodes without edges). A matching is a set of edges not incident to each other. Show that the maximum matching problem in a bipartite graph can be seen as a maximum flow problem. To do so, give the input and the output of this flow problem and explain how to retrieve the matching from the flow.

▶ **Correction**

Let there be a bipartite graph $G = (V \cup W, A)$, we add a source s and a sink t, connecting s to V and W to t. All edges have capacity 1. Thus, the incoming flow to each node in V is 1. The outgoing flow from each node in W is 1. By the conservation constraint, there can be at most one outgoing edge from a node in V entering a node in W. Therefore, the edges between V and W carrying flow form a matching whose size is equal to the value of the flow. Conversely, if we have a matching M, then we can assign a flow of 1 on all these edges and on the edges (s, v) such that $(v, w) \in M$ and on all the edges (w, t) such that $(v, w) \in M$. We then obtain a valid flow whose value is equal to $|M|$. Thus, finding a maximum flow in this graph is indeed equivalent to finding a maximum matching.

Exercice 5 — *Wedding dinner*

A couple wants to organize its wedding dinner. In order to increase social interactions, the married ones do not want to place two friends or members of a same family at the same table. How this problem could be modelized by a maximum flow problem ? We consider there are p groups of friends/families, the i-th group contains $a(i)$ members. There are q tables and $b(j)$ seats at the j-th table.

▶ **Correction**

This exercise is similar to the previous one, where we are looking for a weighted matching (each edge of the weighted matching can touch a certain number of other edges). We create a graph $G = (V \cup W, A)$ where each vertex in V is a group and each vertex in W is a table. We add a source s and a sink t. We connect s to V and W to t. The first edges have a capacity of $a(i)$ and the others have a capacity of $b(j)$. We connect every node in V to every node in W with an edge of capacity 1.

Exercice 6 — *Experiment choice*

We want to choose scientific experiments among n experiments $E_1, \ldots, E_n$ that must be done in outer space. Each experiment needs tools that should be chosen among p tools $I_1, \ldots, I_p$. Each tool may be used by multiple experiments. By conducting the experiment E_i, we earn p_i euros, but carrying the tool I_j into outer space costs c_j euros. The objective is to maximize the benefits.

We are going to modelize this problem with the minimum cut problem in the following transportation network :

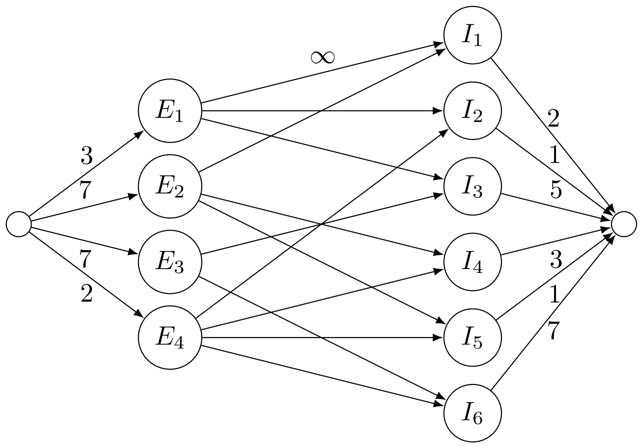
For each experiment E_i, we add a node E_i. For each tool I_j, we add a node I_j. There is also a source s and a sink t. An arc of capacity p_i goes from s to E_i and an arc of capacity c_j goes from I_j to t. Finally, if the experiment E_i uses the tool I_j, then there is an arc of infinite capacity from E_i to I_j.

1. What is the network associated to the following experiments ? Find a minimum cut in that network.

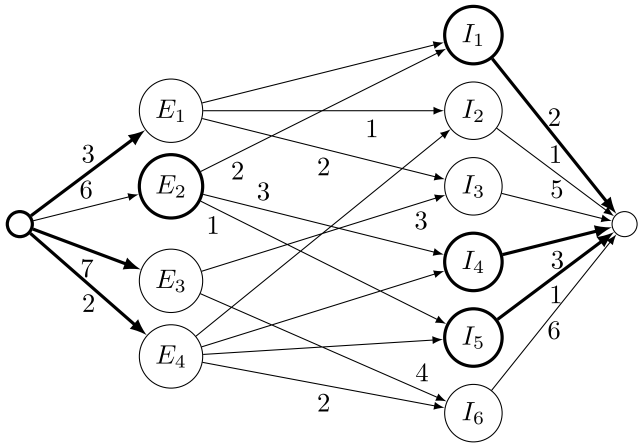
	p_i	I_1	I_2	I_3	I_4	I_5	I_6
c_j		2	1	5	3	1	7
E_1	3	x	x	x			
E_2	7	x			x	x	
E_3	7			x			x
E_4	2		x		x	x	x

2. Show that any feasible solution may be associated with a cut separating s from t with finite capacity. What is the link between the capacity of that cut and the benefits of the solution ?
3. Deduce how to solve the example of question 1.

► **Correction**

1. 

   We obtain the flow and the following cut (with S in bold), of capacity 18.

2. Let's consider a cut (S, T) of finite capacity. Thus, in this cut, if $E_i \in S$ then, for every instrument I_j used by E_i ; $I_j \in S$ (otherwise the capacity of the cut is infinite). Likewise, by contraposition, if $I_j \in T$ then for every experiment E_i using I_j, we have $E_i \in T$.

   Suppose we perform all experiments such that $E_i \in S$ and bring all instruments such that $I_j \in S$. Then the gain B is

$$B = \sum_{E_i \in S} (p_i) - \sum_{I_j \in S} (c_j)$$

$$C(S,T) = \sum_{E_i \in T} (p_i) + \sum_{I_j \in S} (c_j)$$

$$B = \sum_{E_i} (p_i) - C(S,T)$$

3. It is therefore necessary to carry out the experiment E_2 with the instruments I_1, I_4, and I_5, for a gain of 1.