Tutorial 7 : Markov chains

Operations research, 3rd semester.

2024

Exercice 1 — *Rock Paper Scissors Lizard Spock*

The rules of Rock Paper Scissors adapted by Dr Sheldon Cooper are the following : scissors cuts paper, paper covers rock, rock crushes lizard, lizard poisons Spock, Spock smashes scissors, scissors decapitates lizard, lizard eats paper, paper disproves Spock, Spock vaporizes rock and, as it always has, rock crushes scissors.

A tournament is about to start. A player wins a versus battle if he is the first to win 100 rounds. One of the players, Leonard, studied one of his opponents, Howard. He deduced the following rules :

— when Hower plays Rock, he then plays uniformly one of the other shapes ;

— when he plays Lizard, he then plays Lizard again ;

— when he plays Paper, he then plays Scissors ;

— when he plays Spock, he then plays Lizard or Spock, he plays Lizard four times more than Spock ;

— finally, when he plays Scissors, he then plays Rock, Spock or Lizard, he plays Spock twice more than Lizard and he plays Rock twice more than Lizard too.

We can modelize the way Howard plays by a Markov chain.

1. What is the stochastic process $\{X_t \in S\}_{t \in T}$ of that chain : what are the states S, the time T and why is this process a Markov chain ?

▶ **Correction**

States : the 5 moves that Howard can play.

$T = \mathbb{N}$, these are the turns of the game.

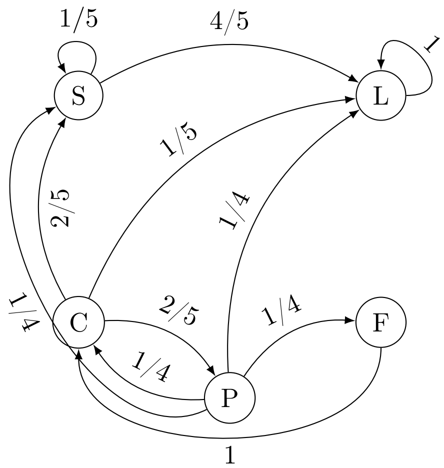
It is a Markov chain because the process is Markovian (Howard determines his next move solely based on the last move he made) and because the process is homogeneous (the probabilities he uses to decide his next move do not change over time).

2. What is the transition matrix and the associated graph ?

▶ **Correction**

$S = \{Stone/P, Paper/F, Scissors/C, Lizard/L, Spock/S\}$, in this order

$$\begin{pmatrix} 0 & 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 0 & 1 & 0 & 0 \\ 2/5 & 0 & 0 & 1/5 & 2/5 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 4/5 & 1/5 \end{pmatrix}$$



3. What are the communicating class ? Is the chain irreducible ?

▶ **Correction**

{CPF}, {S} and {L}. Not irreducible because 3 classes instead of one.

4. Which states are transient, recurrent or absorbing ?

▶ **Correction**

All transitory except L. L is recurrent and absorbing.

5. What is the probability that Howard plays Rock 3 rounds after playing Rock ? Lizard 3 rounds after playing Spock ?

▶ **Correction**

We have a circuit of size 3 : PFCP. So the probability is only 1/4 * 1 * 2/5 = 1/10.
For S -> L : SSSL, SSLL, SLLL : 1/5*1/5*4/5 + 1/5 * 4/5 * 1 + 4/5 = 0.032+0.16+0.8=0.992
(Note : It works because all probabilities are independent, so Pr(SSSL or SSLL or SLLL) = P(SSSL) + P(SSLL) + P(SLLL).

6. What are the 2-transitions probabilities ? (the probability of playing i 2 rounds after j for all i and j).

▶ **Correction**

It is necessary to calculate the square of the transition matrix.

$$P^2 = \begin{pmatrix} 0.1 & 0. & 0.25 & 0.5 & 0.15 \\ 0.4 & 0. & 0. & 0.2 & 0.4 \\ 0. & 0.1 & 0.1 & 0.62 & 0.18 \\ 0. & 0. & 0. & 1. & 0. \\ 0. & 0. & 0. & 0.96 & 0.04 \end{pmatrix}$$

7. We assume Howard plays his first turn uniformly, after how many turn Howard has a 1 in 2 chance playing Lizard ? 8 in 10 chances playing Lizard ?

▶ **Correction**

We set $Q(0) = (0.2, 0.2, 0.2, 0.2, 0.2)$.
$Q(1) = Q(0) \cdot P = (0.08, 0.05, 0.25, 0.45, 0.17)$.
$Q(2) = Q(0) \cdot P^2 = (0.1, 0.02, 0.07, 0.656, 0.154)$.
$Q(3) = Q(0) \cdot P^3 = (0.028, 0.025, 0.045, 0.8182, 0.0838)$.
It takes 2 turns for Lizard to play with a 1 in 2 chance and 3 turns for it to play with an 8 in 10 chance.

8. How Leonard must choose his next shape in order to maximize his chances of winning ? to minimize his chance of loosing ?

▶ **Correction**

We only rely on the transition matrix. For example, if he plays Rock, we know he will play Scissors, so we should play Spock or Rock to win, or Scissors to avoid losing.

If he plays Spock, he will play Spock (1/5) or Lizard (4/5). With a one in five chance, he should play Paper or Lizard to win against Spock, or four times out of five, Rock or Scissors to win against Lizard. He can also play Spock or Lizard to avoid losing.

9. Is the chain regular ? If so, what is the stationary distribution ? Deduce how Leonard must play in order to win against Howard.

▶ **Correction**

One class of stationary states : $\{L\}$. The state of this class is aperiodic. Thus, the chain is regular.

It converges to $(0, 0, 0, 1, 0)$ (this is trivial, but we can still perform the calculation to be convinced).

$$\begin{aligned}
Q &= QP \\
q_1 &= 2/5q_3 \\
q_2 &= 1/4q_1 \\
q_3 &= 1/4q_1 + q_2 \\
q_4 &= 1/4q_1 + 1/5q_3 + q_4 + 4/5q_5 \\
q_5 &= 1/4q_1 + 2/5q_3 + 1/5q_5 \\
q_1 + q_2 + q_3 + q_4 + q_5 &= 1
\end{aligned}$$

Exercice 2 — ***Advertising***

There are three competiting products P_1, P_2 and P_3. We know that 30% of surveyed people prefer P_1, 350% prefer P_2 and the rest prefer P_3. Using an advertising campaign, the society selling P_1 tries to increase its market share. After the campaign, we analyze which clients changed their minds :

before \ after	P_1	P_2	P_3
P_1	50%	40%	10%
P_2	30%	70%	0%
P_3	20%	0%	80%

For example, we see that 20% of the people who prefered P_2 now prefer P_1.

We can modelize the campaign effect by a Markov chain.

1. What is the stochastic process $\{X_t \in S\}_{t \in T}$ of that chain : what are the states S, the time T and why is this process a Markov chain ?
2. What is the transition matrix and the associated graph ?
3. What are the market shares after the campaign ?

▶ **Correction**

$Q(1) = Q(0) \cdot M$ where M is the transition matrix, and $Q(0)$ is the market state before the campaign : $(0.3, 0.5, 0.2)$. So $Q(1) = (0.34, 0.47, 0.19)$.

4. We redo the same campaign, we assume the effects are the same. Give, for each product P, how many percents of people of preferred P before the first campaign are now preferring P_1, P_2 or P_3.

▶ **Correction**

We request the square of the transition matrix.

5. What are the market shares after the second campaign ?

▶ **Correction**

$Q(2) = Q(1) \cdot M = (0.349, 0.465, 0.186)$.

6. We assume the campaign is redone indefinitely. Does a market shares limit exist ? In that case, what is it ?

▶ **Correction**

The chain is irreducible and aperiodic : therefore, it is regular. $Q^* = (6/17, 8/17, 3/17)$.

Exercice 3 — *Work policy*

A public work society has a team where every member works on the same work site. It can work on 2 kinds of work sites : medium works (1 week, type A), or long works (2 weeks, type B). Statistically, every Monday, the society receives with a 1 in 2 chance a request of type A, and with a 3 in 5 chances a request of type B. The requests are independent : the society can receive a request of type A and a request of type B the same week. In that case, the team always choose the type B request. The team cannot work on two sites at the same time : if it is working on a type B work site and receives a request, it is ignored.

In case of a type A work, the society receives 500 euros. It earns 1200 euros at the end of a type B work. Finally, it looses 250 euros every inactive week.

We can modelize the activity every week by a Markov chain.

1. What is the stochastic process $\{X_t \in S\}_{t\in T}$ of that chain : what are the states S, the time T and why is this process a Markov chain ?

▶ **Correction**

It can be modeled with 4 states : one inactive week (State I), one week A (State A), one first week B (State B1), and one second week B (State B2). It is also possible to think of modeling it with 2 states : the beginning of an inactive week or the beginning of an active week (type B), but then we do not differentiate between weeks A and inactive weeks, which does not allow us to calculate the average gain.

The time is weekly : $T = \mathbb{N}$.

It is a Markov chain because each subsequent state depends only on the previous state and the probabilities do not depend on time.

2. What is the transition matrix and the associated graph ?

▶ **Correction**

3 types of probabilities : $p_A = (0.5)(1 - 0.6)$ when we receive a type A and no type B, $p_B = (0.6)$ when we receive a type B, and $p_R = (1-0.5)(1-0.6)$ when we receive no request.

Before \ After	I	A	B1	B2
I	pR	pA	pB	0
A	pR	pA	pB	0
B1	0	0	0	1
B2	pR	pA	pB	0

3. What are the communicating class ? Is the chain irreducible ?

▶ **Correction**

1 strongly connected component, irreducible

4. Which states are transient, recurrent or absorbing ?

▶ **Correction**

All recurrent, no absorbing

5. We assume the society works indefinitely. Does a stationary distribution exist ? In that case, what is the mean profit every week ?

▶ **Correction**

1 recurrent class and all states are aperiodic (2 loops for I and A, and B1 and B2 are in cycles of size 2 and 3). Therefore, regular chain.

We calculate the stationary distribution and find $Q^* = (1/8, 1/8, 3/8, 3/8)$.

The expectation is $Q^* \cdot (-250, 500, 600, 600) = 481.25$ euros.

6. Should the society choose the type A work instead of the type B work when the two requests simultaneously occurs ?

▶ **Correction**

Probabilities change :

3 types of probabilities : pA = (0.5) when receiving type A, pB = (0.6)(1 - 0.5) when receiving type B and not type A, and pR = (1 - 0.5)(1 - 0.6) when not receiving a request.

The chain remains the same, it remains regular.

We calculate the limiting distribution and find $Q^* = (2/13, 5/13, 3/13, 3/13)$.

The expectation is $Q^* \cdot (-250, 500, 600, 600) = 430.77$ euros.

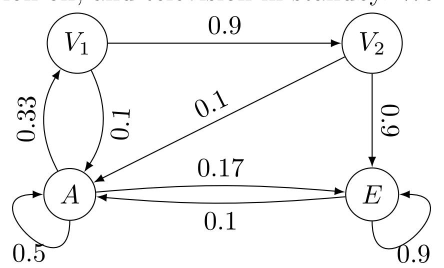
It's less, so it's not as good.

Exercice 4 — *Energy consumption*

A familly uses their television this way : when they look at the television, more or less one hour after, they have 50% chances of continuing looking at it, 33% chances of putting it in sleep mode. The last case consists in shutting it down. After 2h of sleep mode, the TV automatically stops. Every hour, there is 10% chances that someone starts the TV. It consumes 28Wh per hour when it is switched on and 2.5Wh per hour when it is in sleep mode. What it the mean annual consumption in Wh of this television ?

▶ **Correction**

This behavior can be modeled with a Markov chain. There are three states : television on, television off, and television in standby. We would then have the following graph :



where A, V_1, V_2, E are respectively the states On, Sleep, Sleep after 1 hour, and Off. It can be seen that this chain is regular, it has a single class of recurrent states and has loops, so the states are aperiodic. We can calculate its limiting distribution, we obtain

$$Q^* = (0.167, 0.055, 0.050, 0.729)$$

Thus, from the moment the chain has converged to its limiting distribution, we have on average, each hour, 0.167h where the TV is on and 0.105h where the TV is in standby. So each hour, we consume (28*0.167 + 0.105*2.5) Wh, which is approximately 4.939. Therefore, each year, we have approximately 43260 Wh consumed each year.

Exercice 5 — *Festivities*

A big event is organized. The place is cut in 5 zones :

— the entrance E where people come, pay and leave.

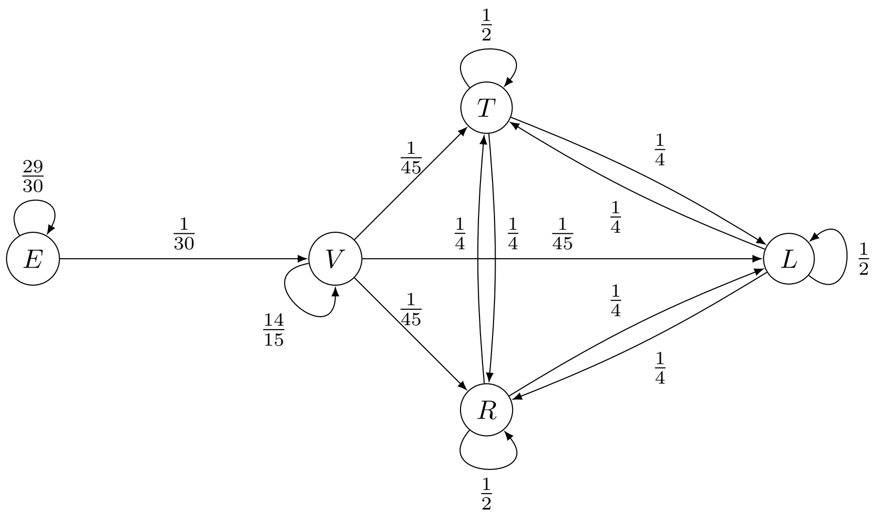
— the cloackroom V where people put their coats.

— Three main zones, the transporation zone T, the logistics zone L and the network zone R.

1. According to some preliminary studies, we know that every minut,
   — Every visitor leave the entrance for the cloackroom with a probability $\frac{1}{30}$. Otherwise they stay in the entrance.
   — Every visitor leave the cloackroom with a probability $\frac{1}{15}$. In that case, they go to the main zones with the same probvability. Otherwise they stay in the cloackroom.
   — A visitor in the zone T has one chance over two to stay, on chance over four to go in R and one chance over four to go in L.
   — A visitor in the zone R has one chance over two to stay, on chance over four to go in T and one chance over four to go in L.
   — A visitor in the zone L has one chance over two to stay, on chance over four to go in T and one chance over four to go in R.

   Use a markov chain to model the movements of the people. Do the graphical representation of the chain.

▶ **Correction**



2. What are the transient and recurrent states ?

▶ **Correction**

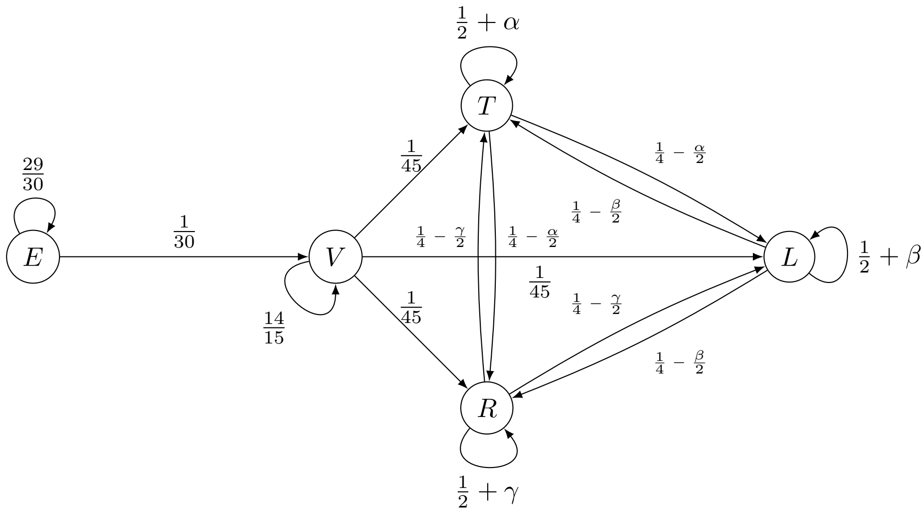
The transient states are E and V, indeed we can leave them for T, L or R but we cannot come back

The recurrent states are R, L and L. Indeded, we can leave those states only to go to the same three states.

3. We would like to add food stands in the zones T, R and L. Every stand increase by 0.1 the probability of staing in the zone, and reduces the probability to go to the each of the other zones by 0.05. We set A (respectively B and C) the number of stands in the zone T (respectively L and R). We want to study how the chain evolve as a function of A, B and C. In order to simplify the calculations, we set $\alpha = 0.1A$; $\beta = 0.1B$ and $\gamma = 0.1C$; then for instance α is exactly the increase of the probability of staying in T.

   Change the chain of question 1 to introduce the values α, β and γ.

▶ **Correction**



4. Show that, if among α, β and γ, at least two values are equal to $\frac{1}{2}$, there is no stationary distribution

▶ **Correction**

Indeed, if for instance $\alpha = \beta = \frac{1}{2}$, then R is transient and T and L do not communicate anymore. There are two recurrent communicating classes so no stationary distribution.

5. What is the stationary distribution if $\alpha = \frac{1}{2}$ and $\beta, \gamma \neq \frac{1}{2}$.

▶ **Correction**

If $\alpha = \frac{1}{2}$ and $\beta, \gamma \neq \frac{1}{2}$ then the stationary distribution is $Q^* = \begin{pmatrix} E & V & T & R & L \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$.

6. In the case where $\alpha, \beta, \gamma \neq \frac{1}{2}$, show that the stationary distribution as a function of α, β and γ is $(q_E^* = 0, q_V^* = 0, q_T^*, q_L^*, q_R^*)$ with

$$q_T^* = \frac{1-2\beta}{1-2\alpha} q_L^* \qquad q_L^* = \frac{1-2\alpha}{1-2\beta} q_T^* \qquad q_R^* = \frac{1-2\alpha}{1-2\gamma} q_T^*$$
$$q_T^* = \frac{1-2\gamma}{1-2\alpha} q_R^* \qquad q_L^* = \frac{1-2\gamma}{1-2\beta} q_R^* \qquad q_R^* = \frac{1-2\beta}{1-2\gamma} q_L^*$$

Et en déduire que

$$q_T^* = \frac{1}{1 + \frac{1-2\alpha}{1-2\beta} + \frac{1-2\alpha}{1-2\gamma}}$$
$$q_L^* = \frac{1}{1 + \frac{1-2\beta}{1-2\alpha} + \frac{1-2\beta}{1-2\gamma}}$$
$$q_R^* = \frac{1}{1 + \frac{1-2\gamma}{1-2\beta} + \frac{1-2\gamma}{1-2\alpha}}$$

▶ **Correction**

We have $q_E^* = q_V^* = 0$ as those states are transient.

$$
\begin{aligned}
q_T^* &= (\frac{1}{2}+\alpha)q_T^* + (\frac{1}{4}-\frac{\gamma}{2})q_R^* + (\frac{1}{4}-\frac{\beta}{2})q_L^* \\
q_T^*(\frac{1}{2}-\alpha) &= (\frac{1}{4}-\frac{\gamma}{2})q_R^* + (\frac{1}{4}-\frac{\beta}{2})q_L^* \\
q_T^* &= \frac{2}{1-2\alpha}\left((\frac{1}{4}-\frac{\gamma}{2})q_R^* + (\frac{1}{4}-\frac{\beta}{2})q_L^*\right) \\
q_T^* &= \frac{1}{1-2\alpha}\left((\frac{1}{2}-\gamma)q_R^* + (\frac{1}{2}-\beta)q_L^*\right)
\end{aligned}
$$

However,

$$
\begin{aligned}
q_L^* &= (\frac{1}{2}+\beta)q_L^* + (\frac{1}{4}-\frac{\gamma}{2})q_R^* + (\frac{1}{4}-\frac{\alpha}{2})q_T^* \\
q_L^*(\frac{1}{2}-\beta) &= (\frac{1}{4}-\frac{\gamma}{2})q_R^* + (\frac{1}{4}-\frac{\alpha}{2})\frac{1}{1-2\alpha}\left((\frac{1}{2}-\gamma)q_R^* + (\frac{1}{2}-\beta)q_L^*\right) \\
q_L^*(\frac{1}{2}-\beta) &= (\frac{1}{4}-\frac{\gamma}{2})q_R^* + \frac{1}{4}\left((\frac{1}{2}-\gamma)q_R^* + (\frac{1}{2}-\beta)q_L^*\right) \\
q_L^*(2-4\beta) &= (1-2\gamma)q_R^* + \left((\frac{1}{2}-\gamma)q_R^* + (\frac{1}{2}-\beta)q_L^*\right) \\
q_L^*(\frac{3}{2}-3\beta) &= (\frac{3}{2}-3\gamma)q_R^* \\
q_L^*(1-2\beta) &= (1-2\gamma)q_R^*
\end{aligned}
$$

By symetry

$$
\begin{aligned}
q_L^* &= \frac{1-2\gamma}{1-2\beta}q_R^*; \qquad & q_T^* &= \frac{1-2\gamma}{1-2\alpha}q_R^* \\
q_L^* &= \frac{1-2\alpha}{1-2\beta}q_T^*; \qquad & q_R^* &= \frac{1-2\alpha}{1-2\gamma}q_T^* \\
q_T^* &= \frac{1-2\beta}{1-2\alpha}q_L^*; \qquad & q_R^* &= \frac{1-2\beta}{1-2\gamma}q_L^*
\end{aligned}
$$

Finally, as $q_L^* + q_R^* + q_T^* = 1$

$$
\begin{aligned}
q_R^* &= \frac{1}{1+\frac{1-2\gamma}{1-2\beta}+\frac{1-2\gamma}{1-2\alpha}} \\
q_T^* &= \frac{1}{1+\frac{1-2\alpha}{1-2\beta}+\frac{1-2\alpha}{1-2\gamma}} \\
q_L^* &= \frac{1}{1+\frac{1-2\beta}{1-2\alpha}+\frac{1-2\beta}{1-2\gamma}}
\end{aligned}
$$

7. What should be the values of A, B and C is we want to have twice more people in T and L and R after some time ?

▶ **Correction**

We can assume that the stationary distribution is aeached after some time.
We want $q_T^* = \frac{1-2\beta}{1-2\alpha}q_L^* = 2q_L^*$, then $1-2\beta = 2-4\alpha$. Similarly $1-2\gamma = 2-4\alpha$.
So $\alpha = \frac{1}{4}+\frac{\beta}{2} = \frac{1}{4}+\frac{\gamma}{2}$.
Thus $A = 2.5 + \frac{B}{2} = 2.5 + \frac{C}{2}$.
We can then use

- $A = 3$ et $B = C = 1$.
- $A = 4$ et $B = C = 3$

Be careful as $A = B = C = 5$ is not a solution : there would not be any stationary distribution.