Tutorial 8 : Queuing

Operations research, 3rd semester.

2024

Exercice 1 — *File d'attente (6 points)*

We consider a queue. For each of the following cases, where we describe the birth rate λ_n in arrival per second and the death rate μ_n in leaving per second par seconde if there exists a stationary distribution. In that case, what is the value P_n for every $n \geq 0$. We recall thatexp$(x) = \sum_{n=0}^{+\infty} \frac{x^n}{n!}$.

1. $\lambda_n = 3$, $\mu_n = 5$

▶ **Correction**

If λ_n and μ_n are constant with $\lambda_n < \mu_n$, there is a stationary distribution. We then have $P_0 = 1 - \frac{3}{5}$ and $P_n = (1 - \frac{3}{5}) \cdot (\frac{3}{5})^n$.

2. $\lambda_n = 5$, $\mu_n = 3$

▶ **Correction**

If λ_n and μ_n are constant with $\lambda_n > \mu_n$, there is no stationary distribution.

3. $\lambda_n = (n+1)$, $\mu_n = 100$

▶ **Correction**

There is a stationnary distribution if and only ifthe serie $\sum_{n=1}^{+\infty} \frac{\lambda_0 \cdot \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \cdot \mu_2 \cdots \mu_n}$ converges. However, $\frac{\lambda_0 \cdot \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \cdot \mu_2 \cdots \mu_n} \simeq_{n \to +\infty} \frac{n!}{100^n}$. This sequence does not converge toward 0 thus the serie does not converge. There is no stationary distribution.

4. $\lambda_n = (n+1)$, $\mu_n = n^2$

▶ **Correction**

There is a stationary distribution if and only if $\sum_{n=1}^{+\infty} \frac{\lambda_0 \cdot \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \cdot \mu_2 \cdots \mu_n}$ converges. In this case $\frac{\lambda_0 \cdot \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \cdot \mu_2 \cdots \mu_n} = \frac{n!}{n!^2} = \frac{1}{n!}$. This serie converges toward $exp(1) - 1$.

We then have $P_0(1 + \sum_{n=1}^{+\infty} \frac{\lambda_0 \cdot \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \cdot \mu_2 \cdots \mu_n}) = 1$

$P_0 = 1/e$

$P_n = \frac{1}{n!} 1/e$.

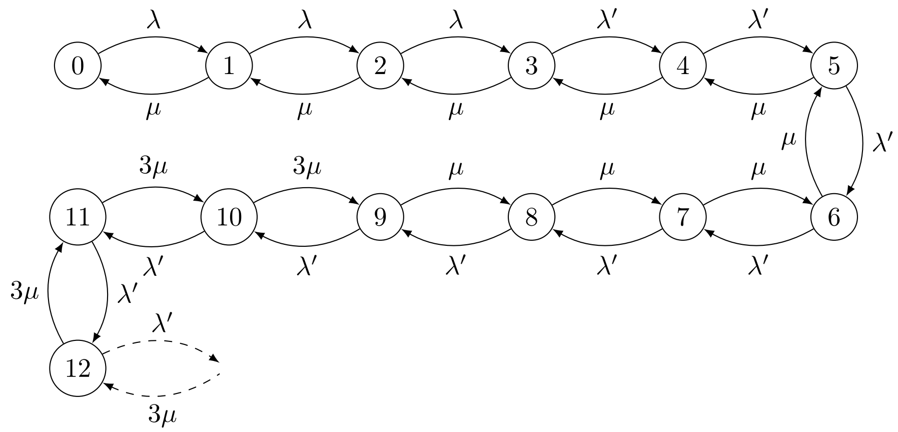
Exercice 2 — *Supermarket queue*

In a supermarket, 5 clients can go through a checkout every 10 minutes. While there are 2 clients or less in the queue, 2 clients come every 5 minutes. After that, 15 clients appear every 20 minutes. If there are 9 clients or less, only one checkout is open. If there are more than 10 clients, 2 more checkouts open.

1. What is the graphical representation of that queue ?

▶ **Correction**

We use the same time unit. We use in this case intervals of 10 minuts.
We set $\mu = 5$, $\lambda = 4$ and $\lambda' = 7.5$.



2. What is the value of $P'_n(t)$ for every n as a function of $P_m(t)$ for every $m \in \mathbb{N}$?

▶ **Correction**

$$\begin{aligned} P_0(t+dt) &= P_0(t) \cdot (1 - \lambda dt) + P_1(t) \cdot \mu dt + o(dt) \\ \Rightarrow P'_0(t) &= -\lambda P_0(t) + \mu P_1(t) \end{aligned}$$

For $i \in [\![1;2]\!]$,

$$\begin{aligned} P_i(t+dt) &= P_{i-1}(t) \cdot \lambda dt + P_i(t) \cdot (1 - (\lambda + \mu)dt) + P_{i+1}(t) \cdot \mu dt + o(dt) \\ \Rightarrow P'_i(t) &= \lambda P_{i-1}(t) - (\lambda + \mu) P_i(t) + \mu P_{i+1}(t) \end{aligned}$$

For $i = 3$,

$$\begin{aligned} P_i(t+dt) &= P_{i-1}(t) \cdot \lambda dt + P_i(t) \cdot (1 - (\lambda' + \mu)dt) + P_{i+1}(t) \cdot \mu dt + o(dt) \\ \Rightarrow P'_i(t) &= \lambda P_{i-1}(t) - (\lambda' + \mu) P_i(t) + \mu P_{i+1}(t) \end{aligned}$$

For $i \in [\![4;8]\!]$,

$$\begin{aligned} P_i(t+dt) &= P_{i-1}(t) \cdot \lambda' dt + P_i(t) \cdot (1 - (\lambda' + \mu)dt) + P_{i+1}(t) \cdot \mu dt + o(dt) \\ \Rightarrow P'_i(t) &= \lambda' P_{i-1}(t) - (\lambda' + \mu) P_i(t) + \mu P_{i+1}(t) \end{aligned}$$

For $i = 9$,

$$\begin{aligned} P_i(t+dt) &= P_{i-1}(t) \cdot \lambda' dt + P_i(t) \cdot (1 - (\lambda' - \mu)dt) + P_{i+1}(t) \cdot 3\mu dt + o(dt) \\ \Rightarrow P'_i(t) &= \lambda' P_{i-1}(t) - (\lambda' + \mu) P_i(t) + 3\mu P_{i+1}(t) \end{aligned}$$

For $i \geq 10$,

$$\begin{aligned} P_i(t+dt) &= P_{i-1}(t) \cdot \lambda' dt + P_i(t) \cdot (1 - \lambda' - 3\mu)dt + P_{i+1}(t) \cdot 3\mu dt + o(dt) \\ \Rightarrow P'_i(t) &= \lambda' P_{i-1}(t) - (\lambda' + 3\mu) P_i(t) + 3\mu P_{i+1}(t) \end{aligned}$$

3. Prove that a stationary distribution exists.

▶ **Correction**

After some n, the birth and death rates are constant and the former is lower than the latter.

4. What is the value of P_n for every n in the stationary distribution ?

► **Correction**

We start by computing P_i as a function of P_0 with :

$$P_i = \frac{\lambda_0\lambda\cdots\lambda_{n-1}}{\mu_1\mu_2\cdots\mu_n}P_0$$

$$\begin{aligned}
&\text{Pour } i \in [\![1;3]\!], && P_i = \left(\frac{\lambda}{\mu}\right)^i P_0\\
&\text{Pour } i \in [\![4;9]\!], && P_i = \left(\frac{\lambda}{\mu}\right)^3 \left(\frac{\lambda'}{\mu}\right)^{i-3} P_0\\
&\text{Pour } i \geq 10 && P_i = \left(\frac{\lambda}{\mu}\right)^3 \left(\frac{\lambda'}{\mu}\right)^6 \left(\frac{\lambda'}{3\mu}\right)^{i-9} P_0
\end{aligned}$$

Or

$$\begin{aligned}
\sum_{i=0}^{+\infty} P_i &= 1\\
P_0 + P_0\sum_{i=1}^{3}\left(\frac{\lambda}{\mu}\right)^i &+\\
P_0\sum_{i=4}^{9}\left(\frac{\lambda}{\mu}\right)^3\left(\frac{\lambda'}{\mu}\right)^{i-3} &+\\
P_0\sum_{i=10}^{+\infty}\left(\frac{\lambda}{\mu}\right)^3\left(\frac{\lambda'}{\mu}\right)^6\left(\frac{\lambda'}{3\mu}\right)^{i-9} &= 1
\end{aligned}$$

Donc

$$\begin{aligned}
P_0(1 + 1.95 + 15.96 + 5.83) &= 1\\
P_0 &= 0.040\\
P_1 &= 0.032\\
P_2 &= 0.025\\
P_3 &= 0.020\\
P_4 &= 0.031\\
P_5 &= 0.047\\
P_6 &= 0.070\\
P_7 &= 0.105\\
P_8 &= 0.157\\
P_9 &= 0.236\\
P_{10} &= 0.11\\
P_{11} &= 0.059\\
P_{12} &= 0.029\\
&\cdots
\end{aligned}$$

We get the following curve, that is coherent with the intuition we have about the number of people on the queue : with less than 2 people, more people leave than the number of arrival, we converge toward 0 ; otherwise below 9 people, more people come, and above 9 people, more people leave then 9 is a stable equilibrium.

5. What is the mean number of people waiting in the queue (and thus not going through a checkout) ?

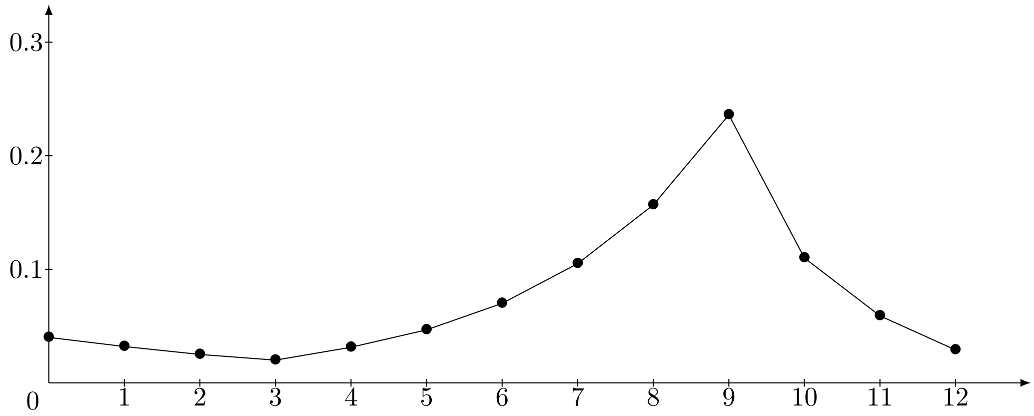


Figure 1 –

▶ **Correction**

We compute

$$\sum_{i=1}^{9}(i-1)\cdot P_i + \sum_{i=10}^{+\infty}(i-3)\cdot P_i$$

$$\begin{aligned}\sum_{i=1}^{9}(i-1)\cdot P_i &= P_0\cdot\left(\sum_{i=1}^{3}(i-1)\left(\frac{\lambda}{\mu}\right)^i\right)+P_0\cdot\left(\sum_{i=4}^{9}(i-1)\left(\frac{\lambda}{\mu}\right)^3\left(\frac{\lambda'}{\mu}\right)^{i-3}\right)\\ &= P_0\cdot(1.66+104.98)=4.31\end{aligned}$$

$$\begin{aligned}\sum_{i=10}^{+\infty}(i-3)\cdot P_i &= \sum_{i=10}^{+\infty}(i-3)\left(\frac{\lambda}{\mu}\right)^3\left(\frac{\lambda'}{\mu}\right)^6\left(\frac{\lambda'}{3\mu}\right)^{i-9}P_0\\ &= \sum_{i=7}^{+\infty} i\left(\frac{\lambda}{\mu}\right)^3\left(\frac{\lambda'}{\mu}\right)^6\left(\frac{\lambda'}{3\mu}\right)^{i-6}P_0\\ &= \left(\frac{\lambda}{\mu}\right)^3\left(\frac{\lambda'}{\mu}\right)^6\left(\frac{\lambda'}{3\mu}\right)^{-5}\cdot\left(\sum_{i=1}^{+\infty} i\left(\frac{\lambda'}{3\mu}\right)^{i-1}P_0-\sum_{i=1}^{6} i\left(\frac{\lambda'}{3\mu}\right)^{i-1}P_0\right)\\ &= \left(\frac{\lambda}{\mu}\right)^3\left(\frac{\lambda'}{\mu}\right)^6\left(\frac{\lambda'}{3\mu}\right)^{-5}\cdot\left(\frac{1}{(1-\lambda'/(3\mu))^2}-\sum_{i=1}^{6} i\left(\frac{\lambda'}{3\mu}\right)^{i-1}P_0\right)\\ &= 186.62\cdot(0.16-0.15)=1.866\end{aligned}$$

Donc

$$\sum_{i=1}^{9}(i-1)\cdot P_i + \sum_{i=10}^{+\infty}(i-3)\cdot P_i = 6.176$$

Exercice 3 — *Interversion of the birth and death rates*

We consider a paradise (for instance, of the pastafarian religion). A dead person comes to the paradise until it is reincarnated and rebirth on Earth.

The mean number of death is 2000 people per second in the world. Those numbers do not depend on the number of people in the paradise.

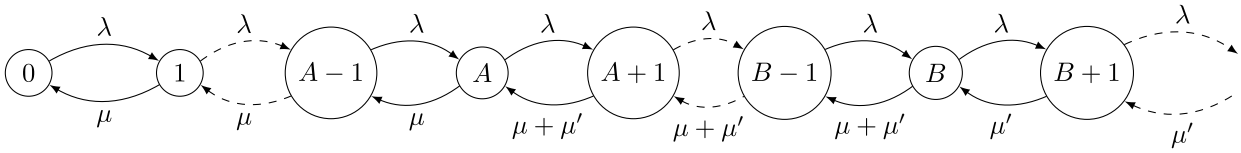
At the reincarnation service, two pirates are in charge : Blackbeard and Barbarossa. Blackbeard reincarnates 60000 people per minutes on average and works only if there are 10000 people or less in the paradise. Barbarossa reincarnates 180000 people on average and works only if there are 5001 people or more in the paradise.

Let $\lambda = 2000$, $\mu = 1000$, $\mu' = 3000$, $A = 5000$ and $B = 10000$.

1. Draw the graphical representation of the queue process associated with the deaths and reincarnations in the Pastafarian paradise. Use the notations λ, μ, μ', A et B.

▶ **Correction**

Be careful with the *birth* and *death* rate which are here respectivement a *dead entering the paradise* rate and a *reincarnation* rate. By setting every rate in people per second, we get μ, $\mu+\mu'$ and μ' as the reincarnation rate when there are respectively at most A, between $A+1$ and B, and at least $B+1$ people in the queue.



2. If we assume that there exists a stationary distribution, why can we assume that this distribution is reached ?

▶ **Correction**

We can assume that people have lived and died for a very long time. The stationary distribution is then reached.

3. Show that the stationnary distribution exists.

▶ **Correction**

After some value of n, the birth and deat rates are equals to λ and μ'. As $\lambda < \mu'$, there is a stationary distribution.

4. Let P_n be the probability that there are n people in the paradise. Describe, as a function of λ, μ, μ', A, B, P_0 and n, the probability P_n. Using the numerical values, obtain P_n as a function of A, P_0 and n.

▶ **Correction**

For all $1 \leq n \leq A, P_n = \left(\frac{\lambda}{\mu}\right)^n P_0 = 2^n P_0$.

For all $A+1 \leq n \leq B = 2A, P_n = \left(\frac{\lambda}{\mu}\right)^A \left(\frac{\lambda}{\mu+\mu'}\right)^{n-A} P_0 = 2^A \left(\frac{1}{2}\right)^{n-A} P_0 = 2^{2A}\frac{1}{2^n} P_0$.

For all $B+1 = 2A+1 \leq n, P_n = \left(\frac{\lambda}{\mu}\right)^A \left(\frac{\lambda}{\mu+\mu'}\right)^{B-A} \left(\frac{\lambda}{\mu'}\right)^{n-B} P_0 = 2^A \left(\frac{1}{2}\right)^A \left(\frac{2}{3}\right)^{n-2A} P_0 = \left(\frac{3}{2}\right)^{2A} \left(\frac{2}{3}\right)^n P_0$.

5. Show that $P_0 = 1/\left(2^{A+1} + 2^A\right)$.

▶ **Correction**

Donc

$$P_0 = \frac{1}{1 + \sum_{n=1}^{A} 2^n + \sum_{n=A+1}^{2A} 2^{2A} \frac{1}{2^n} + \sum_{n=2A+1}^{+\infty} \left(\frac{3}{2}\right)^{2A} \left(\frac{2}{3}\right)^n} \tag{1}$$

$$P_0 = \frac{1}{2^{A+1} - 1 + \sum_{n=A+1}^{2A} 2^{2A} \frac{1}{2^n} + \sum_{n=2A+1}^{+\infty} \left(\frac{3}{2}\right)^{2A} \left(\frac{2}{3}\right)^n} \tag{2}$$

$$P_0 = \frac{1}{2^{A+1} - 1 + 2^{2A}(\frac{1}{2^A} - \frac{1}{2^{2A}}) + \sum_{n=2A+1}^{+\infty} \left(\frac{3}{2}\right)^{2A} \left(\frac{2}{3}\right)^n} \tag{3}$$

$$P_0 = \frac{1}{2^{A+1} - 1 + (2^A - 1) + \sum_{n=2A+1}^{+\infty} \left(\frac{3}{2}\right)^{2A} \left(\frac{2}{3}\right)^n} \tag{4}$$

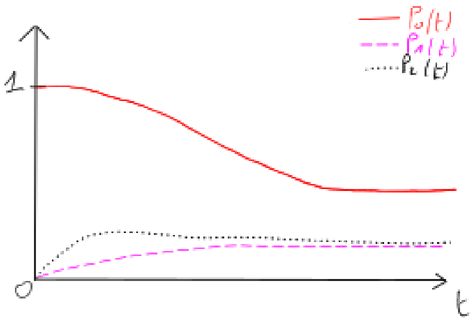
$$P_0 = \frac{1}{2^{A+1} - 1 + (2^A - 1) + \left(\frac{3}{2}\right)^{2A} 3(\frac{2}{3})^{2A+1}} \tag{5}$$

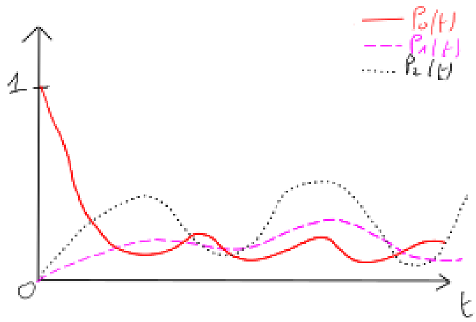
$$P_0 = \frac{1}{2^{A+1} - 1 + (2^A - 1) + 2} \tag{6}$$

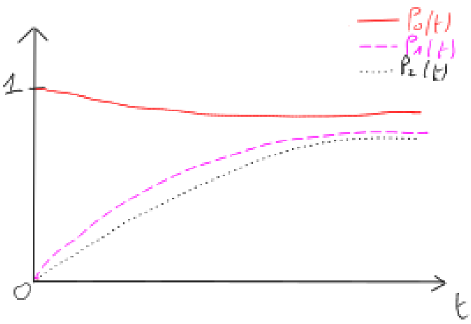
$$P_0 = \frac{1}{2^{A+1} + 2^A} \tag{7}$$

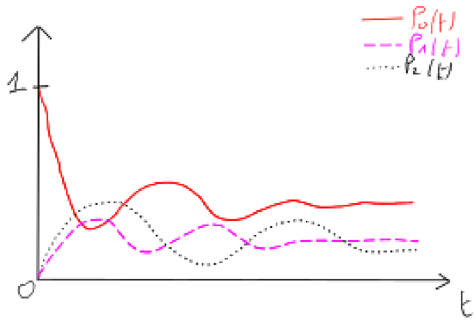
Exercice 4 — *Queues and curves* We consider a queue with a birth rate λ_n and a death rate μ_n. We assume that after some value n, the rates are constant and equal μ et λ, and we assume that $\mu > \lambda$.

We drawn 4 graphics with three curves $P_0(t)$, $P_1(t)$ and $P_2(t)$ depending on t. For each graphic, could the curves correspond to the probability that there are 0, 1 or 2 people in the queue ?









▶ **Correction**

There is a stationary distribution as $\mu > \lambda$ so the curves should converge after some time. This eliminates the upper right curves. The curves on the below left part are not possible as the sum of the probabilities is greater than 1. For the two others, there is no clear counter argument.

Exercice 5 — *Queue of pairs*

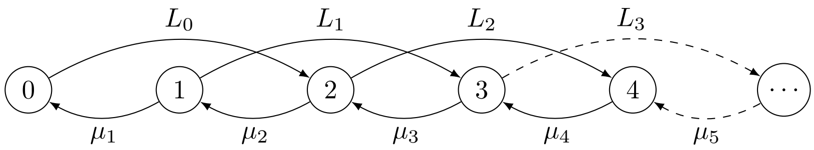
A famous hairdresser is in high demand but does not take appointments. So there is always a long queue. We want to measure it. We realize that there are in fact people going to this hairdresser because he offers discount vouchers for pairs of people. So we must consider that people arrive not one by one in the queue but two by two. On the other hand, they always leave one by one. We therefore rewrite the arrival and departure probabilities as follows :

$$\begin{aligned}
&Pr(X(t+dt)-X(t)=2 & |X(t)=n) &= & L_n dt + o(dt)\\
&Pr(X(t+dt)-X(t)=-1 & |X(t)=n>0) &= & \mu_n dt + o(dt)\\
&Pr(X(t+dt)-X(t)=0 & |X(t)=n>0) &= & (1-L_n-\mu_n)dt + o(dt)\\
&Pr(X(t+dt)-X(t)=0 & |X(t)=0) &= & (1-L_0)dt + o(dt)\\
&Pr(X(t+dt)-X(t)\notin\{-1,0,2\} & |X(t)=n) &= & 0 + o(dt)
\end{aligned}$$

We write $P_n(t)$ the probability that there are n people in the queue at time t and P_n the probability in the stationary distribution if it exists.

1. Do the graphical representation of the queue.

▶ **Correction**



2. Show that $P'_n(t) = P_{n-2}(t) \cdot L_{n-2} - P_n(t) \cdot (L_n + \mu_n) + P_{n+1}(t) \cdot \mu_{n+1}$ if $n > 2$.

▶ **Correction**

We compute $P_n(t + dt)$. At that time there can be n people in the queue in multiple cases :

— there were n people at time t and, with probability $(1 - L_n - \mu_n)dt + o(dt)$, no one arrived.

— there were $n - 2$ people at time t and, with probability $L_{n-2}dt + o(dt)$, two people arrived.

— there were $n + 1$ people at time t and, with probability $\mu_{n+1}dt + o(dt)$, one person left.

— there were another number of people at time t and with a probability $o(dt)$ this value came to n.

$$\begin{aligned}
P_n(t + dt) &= P_{n-2}(t) \cdot (L_{n-2}dt + o(dt)) + P_n(t) \cdot ((1 - L_n - \mu_n)dt + o(dt)) \\
&\quad + P_{n+1}(t) \cdot (\mu_{n+1}dt + o(dt)) + \sum_{\substack{k=0 \\ k \neq n-2 \\ k \neq n \\ k \neq n+1}}^{+\infty} P_k \cdot o(dt) \\
\frac{P_n(t + dt) - P_n(t)}{dt} &= P_{n-2}(t) \cdot L_{n-2} - P_n(t) \cdot (L_n + \mu_n) + P_{n+1}(t) \cdot \mu_{n+1} + \sum_{k=0}^{+\infty} P_k \cdot o(1) \\
\frac{P_n(t + dt) - P_n(t)}{dt} &= P_{n-2}(t) \cdot L_{n-2} - P_n(t) \cdot (L_n + \mu_n) + P_{n+1}(t) \cdot \mu_{n+1} + o(1) \\
\lim_{dt \to 0} \frac{P_n(t + dt) - P_n(t)}{dt} &= P_{n-2}(t) \cdot L_{n-2} - P_n(t) \cdot (L_n + \mu_n) + P_{n+1}(t) \cdot \mu_{n+1} \\
P'_n(t) &= P_{n-2}(t) \cdot L_{n-2} - P_n(t) \cdot (L_n + \mu_n) + P_{n+1}(t) \cdot \mu_{n+1}
\end{aligned}$$

3. What are the formulas of $P'_0(t)$ and $P'_1(t)$.

▶ **Correction**

$$\begin{aligned}
P'_0(t) &= -P_0(t) \cdot L_0 + P_1(t) \cdot \mu_1 \\
P'_1(t) &= -P_1(t) \cdot (L_1 + \mu_1) + P_2(t) \cdot \mu_2
\end{aligned}$$

4. Assuming there exists a stationary distribution, show that $P_1 = \frac{L_0}{\mu_1} P_0$ and $P_2 = (L_1 + \mu_1) \cdot \frac{L_0}{\mu_1 \mu_2} P_0$.

▶ **Correction**

We have $P'_n(t) = 0$ and $P_n(t) = P_n$.

$$0 = -P_0 \cdot L_0 + P_1 \cdot \mu_1$$

Donc

$$P_1 = \frac{L_0}{\mu_1} P_0$$

Et

$$0 = -P_1 \cdot (L_1 + \mu_1) + P_2 \cdot \mu_2$$
$$0 = -(L_1 + \mu_1) \frac{L_0}{\mu_1} P_0 + P_2 \cdot \mu_2$$

Donc

$$P_2 = (L_1 + \mu_1) \cdot \frac{L_0}{\mu_1 \mu_2} P_0$$

We set for $n \geq 0$:

$$NS(n) = \{I \subseteq [\![1;n]\!] | \forall i < j \in I, i+1 \neq j\}$$
$$P(I,n) = \prod_{i \in I} \mu_i \prod_{\substack{i \notin I \\ i \leq n}} L_i$$

For instance $NS(5) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{1,3\}, \{1,4\}, \{1,5\}, \{2,4\}, \{2,5\}, \{3,5\}, \{1,3,5\}\}$ et $P(\{1,4\},4) = \mu_1 L_2 L_3 \mu_4$.

Note that : $NS(0) = \emptyset$, $P(\emptyset, 0) = 1$.

5. Show that, for $n \geq 1$, $\sum\limits_{\substack{I \in NS(n+1) \\ n+1 \in I}} P(I, n+1) = \mu_{n+1} \sum\limits_{\substack{I \in NS(n) \\ n \notin I}} P(I,n)$

   *Clue : **do not** use a proof by induction.*

► **Correction**

Let $I \in NS(n+1)$ such that $n+1 \in I$. Let $J = I \backslash (n+1)$ then $J \in \{J \subseteq [\![1;n]\!] | \forall i < j \in J, i+1 \neq j\} = NS(n)$. And by definition of $NS(n+1)$, $n \notin I$.
Conversely if $J \in NS(n)$ such that $n \notin J$ then $J \cup \{n+1\} \in NS(n+1)$ as, for all $j \in J$, $j < n$ then $j+1 \neq n+1$. Thus the set I of $NS(n+1)$ containing $n+1$ equals the sets J of $NS(n)$ not containing n and to which we added $n+1$.
$\sum\limits_{\substack{I \in NS(n+1) \\ n+1 \in I}} P(I, n+1) = \sum\limits_{\substack{I \in NS(n) \\ n \notin I}} P(I \cup \{n+1\}, n+1) = \mu_{n+1} \sum\limits_{\substack{I \in NS(n) \\ n \notin I}} P(I,n)$.

6. Show that, for $n \geq 1$, $\sum\limits_{\substack{I \in NS(n+1) \\ n+1 \notin I}} P(I, n+1) = L_{n+1} \cdot \sum\limits_{I \in NS(n)} P(I,n)$

   Clue : again, no induction

► **Correction**

Let $I \in NS(n+1)$ such that $n+1 \notin I$. Then $I \in \{I \subseteq [\![1;n+1]\!] | n+1 \notin I$ et $\forall i < j \in I, i+1 \neq j\}$. So $I \in \{I \subseteq [\![1;n]\!] | \forall i < j \in I, i+1 \neq j\}$. And then $I \in NS(n)$. On the other hand, if $I \in NS(n)$ then by definition it does not contain $n+1$.

$$\sum_{\substack{I \in NS(n+1) \\ n+1 \notin I}} P(I, n+1) = \sum_{I \in NS(n)} P(I, n+1)$$
$$= L_{n+1} \cdot \sum_{I \in NS(n)} P(I,n)$$

7. Show that, for $n \geq 1$, $\sum\limits_{I \in NS(n+1)} P(I, n+1) = L_{n+1} \cdot \sum\limits_{I \in NS(n)} P(I,n) + \mu_{n+1} \cdot \sum\limits_{\substack{I \in NS(n) \\ n \notin I}} P(I,n)$

   Clue : and again, no induction

▶ **Correction**

Let $I \in NS(n+1)$, then $n+1 \in I$ or $n+1 \notin I$.

Thus $\sum\limits_{I\in NS(n+1)} P(I,n+1) = \sum\limits_{\substack{I\in NS(n+1)\\ n+1\in I}} P(I,n+1) + \sum\limits_{\substack{I\in NS(n+1)\\ n+1\notin I}} P(I,n+1)$.

We can deduce the result from the two previous questions.

8. Show that, for $n \geq 1$, $P_n = \sum\limits_{I\in NS(n-1)} P(I,n-1)\cdot \frac{L_0}{\prod_{i=1}^{n}\mu_i} P_0$

Clue : use the three previous questions. In this case you can use an induction formula.

▶ **Correction**

For $n=1$, we have $P_1 = \frac{L_0}{\mu_1}P_0 = P(\emptyset,0)\frac{L_0}{\mu_1}P_0 = \sum\limits_{I\in NS(0)} P(I,0)\cdot \frac{L_0}{\prod_{i=1}^{n}\mu_i}P_0$.

For $n=2$, we have $P_2 = (L_1+\mu_1)\cdot\frac{L_0}{\mu_1\mu_2}P_0 = (L_1+\mu_1)\cdot\frac{L_0}{\prod_{i=1}^{n}\mu_i}P_0$. In addition, $NS(1)=\{\emptyset,\{1\}\}$ and $P(\emptyset,1)=L_1$ andt $P(\{1\},1)=\mu_1$. Consequently, we have $(L_1+\mu_1) = \sum\limits_{I\in NS(1)} P(I,1)$.

For $n=3$, we prove similarly that $(L_2L_1+L_2\mu_1+\mu_2L_1)\cdot\frac{L_0}{\mu_1\mu_2\mu_3}P_0$. (We need $n=3$ because the induction will use three steps).

We assume the property is true for every $k\leq n$ and we must check it for $n+1$. We know that $0 = P_{n-2}\cdot L_{n-2} - P_n\cdot(L_n+\mu_n) + P_{n+1}\cdot\mu_{n+1}$. So :

$$P_{n+1} = \frac{1}{\mu_{n+1}}\left(P_n\cdot(L_n+\mu_n) + P_{n-2}\cdot L_{n-2}\right)$$

By induction

$$P_{n+1} = \left((L_n+\mu_n)\cdot\sum_{I\in NS(n-1)} P(I,n-1)\cdot\frac{L_0}{\prod\limits_{i=1}^{n}\mu_i}P_0 + L_{n-2}\cdot\sum_{I\in NS(n-3)} P(I,n-3)\cdot\frac{L_0}{\prod\limits_{i=1}^{n-2}\mu_i}P_0\right)\frac{1}{\mu_{n+1}}$$

$$P_{n+1} = \left((L_n+\mu_n)\cdot\sum_{I\in NS(n-1)} P(I,n-1) - \mu_n\mu_{n-1}L_{n-2}\cdot\sum_{I\in NS(n-3)} P(I,n-3)\right)\frac{L_0P_0}{\prod\limits_{i=1}^{n+1}\mu_i}$$

According to question 7, we have the following equality

$$(L_n+\mu_n)\cdot\sum_{I\in NS(n-1)} P(I,n-1) = \sum_{I\in NS(n)} P(I,n) + \mu_n\cdot\sum_{\substack{I\in NS(n-1)\\ n-1\in I}} P(I,n-1)$$

$$P_{n+1} = \left(\sum_{I\in NS(n)} P(I,n) + \mu_n\cdot\sum_{\substack{I\in NS(n-1)\\ n-1\in I}} P(I,n-1) - \mu_n\mu_{n-1}L_{n-2}\cdot\sum_{I\in NS(n-3)} P(I,n-3)\right)\frac{L_0P_0}{\prod\limits_{i=1}^{n+1}\mu_i}$$

According to question 6, we have the following equality

$$L_{n-2}\cdot\sum_{I\in NS(n-3)} P(I,n-3) = \sum_{\substack{I\in NS(n-2)\\ n-2\notin I}} P(I,n-2)$$

$$P_{n+1} = \left(\sum_{I\in NS(n)} P(I,n) + \mu_n\cdot\sum_{\substack{I\in NS(n-1)\\ n-1\in I}} P(I,n-1) - \mu_n\mu_{n-1}\sum_{\substack{I\in NS(n-2)\\ n-2\notin I}} P(I,n-2)\right)\frac{L_0P_0}{\prod\limits_{i=1}^{n+1}\mu_i}$$

According to question 5, we have the following equality

$$\mu_{n-1} \sum_{\substack{I \in NS(n-2) \\ n-2 \notin I}} P(I, n-2) = \sum_{\substack{I \in NS(n-1) \\ n-1 \in I}} P(I, n-1)$$

$$P_{n+1} = \left(\sum_{I \in NS(n)} P(I, n) + \mu_n \cdot \sum_{\substack{I \in NS(n-1) \\ n-1 \in I}} P(I, n-1) - \mu_n \sum_{\substack{I \in NS(n-1) \\ n-1 \in I}} P(I, n-1) \right) \frac{L_0 P_0}{\prod_{i=1}^{n+1} \mu_i}$$

$$P_{n+1} = \left(\sum_{I \in NS(n)} P(I, n) \right) \frac{L_0 P_0}{\prod_{i=1}^{n+1} \mu_i}$$

By induction the result is true.

9. 60 people arrive every hour and 35 leave every half hour. What is the value of P_8 ?

► **Correction**

We must first compute L_i and μ_i. We have $\mu_i = 2\mu = 70$ and $L = \lambda/2 = 30$ (30 pairs of people arrive every hour).

By programming the above formula, we can compute P_0 with $\left(\sum_{i=1}^{+\infty} \sum_{I \in NS(n-1)} P(I, n-1) \cdot \frac{L_0}{\prod_{i=1}^{n} \mu_i} \right)^{-1}$.

We can approximate the sum with the first 30 values. We get more or less $1/6 = 0.16$.
We then compute $P_8 = \sum_{I \in NS(7)} P(I, 7) \cdot \frac{L_0}{\prod_{i=1}^{8} \mu_i} P_0 = 0.29/6 \simeq 4.8410^{-2}$.

Interesting detail : if we take the classical queue (1 arrival at a time) with $\lambda = 60$ and $\mu = 70$ we get a slightly similar result for P_8, with a precision of 10^{-2}.